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AKCE
INTERNATIONAL JOURNAL
OF
GRAPHS AND COMBINATORICS

AKCE Int. J. Graphs Comb., 10, No. 3 (2013), pp. 337-347

In memoriam: Joan Gimbert (1962-2012)

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Abstract

On the first of March 2012, Joan Gimbert, an excellent colleague and mathematician, passed away at the age of 49 years. His research is devoted to Graph Theory making important contributions mainly focused on the degree/diameter problem and other distance related problems on graphs.

Keywords: Almost Moore digraphs, eccentric digraphs, bipartite and multipartite digraphs, radial Moore graphs.

2000 Mathematics Subject Classification: 05C20, 05C35, 05C50, 05C12.

Joan Gimbert was born in Binefar, near Lleida, Spain. He received the degree in Mathematics from the Universitat de Barcelona (UB) in 1985 and his PhD degree in Mathematics from the Universitat Politècnica de Catalunya (UPC) in 2000, with the PhD thesis supervised by professor Miquel Àngel Fiol and entitled *Applications of the spectral theory to the study of dense digraphs.*

*Partially supported by Spanish Project TIN2010-18978 and by Catalan Grant 2009SGR-442

Joan Gimbert was incorporated into teaching work in 1985. His dedication and concern were constant from the first day. Joan Gimbert joined the Polytechnic School of the Universitat de Lleida (UdL) in the course 1991-92 as a teacher of Computer Science Enginyeering, when the degree had only been running a year. He prepared teaching materials with great effort and perseverance. Joan Gimbert was very active and had plenty of initiatives and projects. Little by little, several research lines were set up with which we felt at ease and from which the Lleida research group on Cryptography and Graphs emerged.

The research area which aroused most interest and enthusiasm in Joan Gimbert was Graph Theory, which allowed him to establish many close friendships, collaborating with researchers around the world and making important contributions mainly focused on the degree/diameter problem and other distance related problems on graphs.

2. Joan Gimbert's contributions to Graph Theory

Joan Gimbert's first steps into Graph Theory were in the study of the degree/diameter problem. This is an optimization problem motivated by the modelization of interconnection networks by means of graphs. In this context, given the values of the maximum degree and the diameter of a graph (digraph), there is a natural bound for the number of vertices of it, called *Moore bound*. Graphs attaining such a bound are referred to as *Moore graphs*. The fact that there are very few of them suggested the relaxation of some of the constraints involving the three parameters. This led to the study of graphs and digraphs with small defect; that is, whose order is close to the unattainable Moore bound.

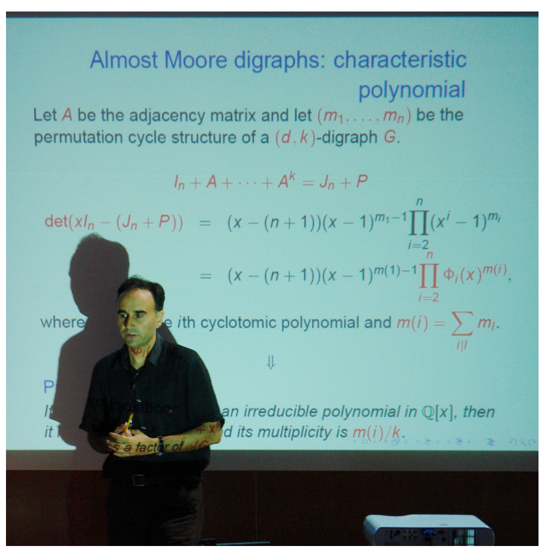


Figure 1: Joan lecturing on digraphs

The classification of such extremal graphs (digraphs), for certain values of the defect and diameter, by using algebraic and spectral techniques, is one of Joan Gimbert's contributions.

Further Gimbert's contributions were related to distance in digraphs. More precisely, he focused on the distance from a particular vertex to the farthest vertex of that one, that is, the *eccentricity*, characterizing the eccentricity sequence of a digraph.

In the rest of the section we present Joan Gimbert's main contributions to Graph Theory. Thanks to the great number of collaborators it was difficult to include all that material. Thus, although we had no intention of giving an exhaustive review of all his publications, our aim was to consider as many of them as possible.

2.1. Almost Moore digraphs

Digraphs having maximum out-degree d, diameter k and whose order is one less than the maximum possible (defect 1), that is, $N = M_D(d,k) - 1$, where

$$M_D(d,k) = 1 + d + d^2 + \cdots + d^k,$$

are called *almost Moore digraphs* or more precisely (d,k)-digraphs. In this context, Miller, Gimbert, Širáň and Slamin proved in [25] that any extremal digraph turns out to be regular. And Joan Gimbert, in his first publication [14], introduced spectral techniques to study the existence and classification of these digraphs.

A (d,k)-digraph G has the property that for each vertex there exists only one vertex, denoted by $r(v)$ and called *repeat* of v (see [1]), such that there are exactly two $v \longrightarrow r(v)$ walks of length less than or equal to k (one of them must be of length k). If $r(v) = v$, which means that v is contained in exactly one k-cycle, v is called *selfrepeat* of v (see Figure 2).

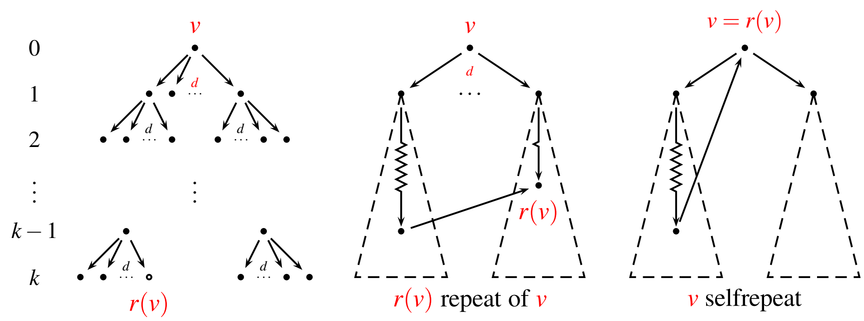


Figure 2: Local structure of a (d,k)-digraph

Seeing r as a permutation of the set of vertices that for each vertex v assigns its repeat $r(v)$, it has a *cycle structure* which corresponds to its unique decomposition in disjoint cycles. Such cycles are called *permutation cycles* of G and the number of permutation cycles of G of each length $i \leq N$ is denoted by m_i.

The $(0,1)$-matrix $P=(p_{ij})$ associated with the permutation r and given by $p_{ij}=1$ if $r(v_i)=v_j$ is related with the adjacency matrix A of G by the equation

$$I+A+A^2+\cdots+A^k=J+P,$$

where J denotes the all-one matrix.

The factorization of $\det(xI-(J+P))$ in $\mathbb{Q}[x]$ in terms of the cyclotomic polynomials $\Phi_n(x)$ is given by (see [2])

$$\det(xI-(J+P))=(x-(N+1))(x-1)^{m(1)-1}\prod_{n=2}^{N}\Phi_n(x)^{m(n)},$$

where $m(n)=\sum_{n|i}m_i$ represents the total number of permutation cycles of order a multiple of n.

The approach that Joan Gimbert followed to search for (d,k)-digraphs was:

- Find the factorization of the characteristic polynomial $\det(xI-A)$ of an almost Moore (d,k)-digraph in terms of its permutation cycle structure $(m_1,\ldots,m_N)$.
- Compute $\operatorname{tr}A,\ldots,\operatorname{tr}A^{k-1}$ from the previous factorization. Obtain conditions for $m_1,\ldots,m_N$ from $\operatorname{tr}A=\cdots=\operatorname{tr}A^{k-1}=0$.
- Construct the set $\mathcal{F}$ of feasible structures. If $\mathcal{F}=\emptyset$, then almost Moore (d,k)-digraphs do not exist.

In this way, Joan Gimbert in his PhD thesis completed the classification of almost Moore digraphs of diameter 2 by studying the factorization of the characteristic polynomial

$$\det(xI-A)=(x-d)(x+1)^{a_1}x^{a_2}(x^2+1)^{b_1}(x^2+2x+2)^{b_2}\prod_{2\le n\le N,\ n\ne 4}(F_{n,2}(x))^{\frac{m(n)}{2}},$$

where

$$F_{n,2}(x)=\Phi_n(1+x+x^2),\quad a_1+a_2=\sum_{1\le n\le N}m_n,\quad b_1+b_2=m(4),$$

and by solving the system determined by $\operatorname{tr}A=\operatorname{tr}A^2=0$. The main results Joan Gimbert obtained were published in [14, 15], showing that line digraph LK_{d+1} of the complete digraph K_{d+1} is the unique $(d,2)$-digraph, if the degree $d\ge 3$. In Figure 3 shows the line digraph LK_5.

For diameters greater than 2, in order to determine the factorization of the characteristic polynomial of the adjacency matrix A, Joan Gimbert proposed the following conjecture concerning the irreducibility of the polynomials $F_{n,k}(x)=\Phi_n(1+x+\cdots+x^k)$:

$$F_{n,k}(x)\text{ is reducible in }\mathbb{Q}[x]\iff\begin{cases}n|(k+2) & \text{if } k \text{ is even}\\ n|2(k+2) & \text{if } k \text{ is odd and } n \text{ is even}\end{cases}$$

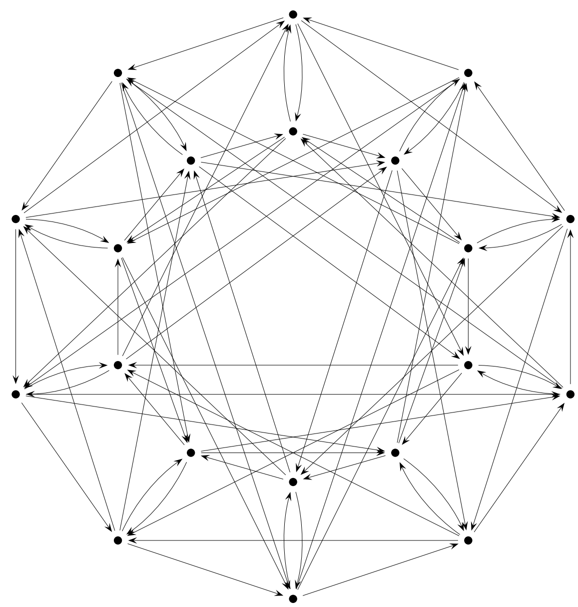


Figure 3: Example of a $(d, 2)$-digraph

in which cases $F_{n,k}(x)$ has just two factors.

Later, in [8], together with Conde, González, Miret and Moreno, his conjecture has been proved for diameter 3 showing as a consequence the nonexistence of almost Moore digraphs of diameter 3.

Concerning the undirected case, the Moore bound is given by

$$M_G(d, k) = 1 + d + d(d-1) + \cdots + d(d-1)^k.$$

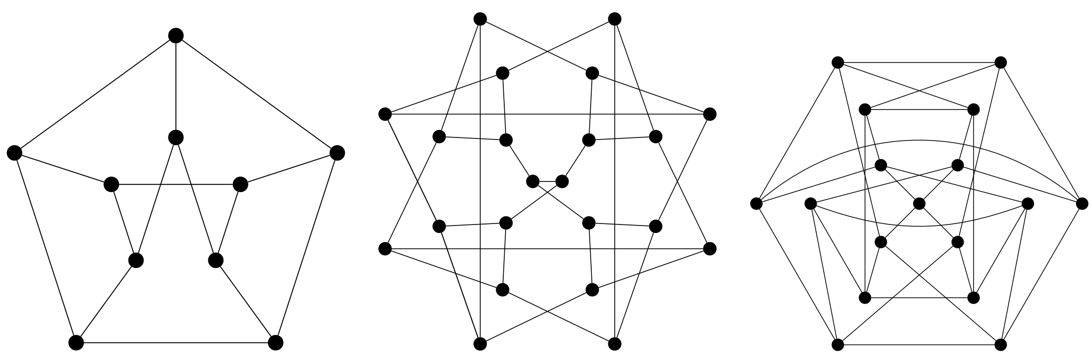
Thus, a graph with maximum degree d and diameter 2 has defect 2 if its order is $d^2 - 1$. Conde and Gimbert in [7] proved, by using algebraic and spectral techniques, that for all values of the degree d within a certain range $4 < d \leq 50$, these graphs with diameter 2 and defect 2 do not exist. Furthermore, they formulated an analogous conjecture to the directed case on the irreducibility of some polynomials in $\mathbb{Q}[x]$ and, assuming this conjecture, they showed the nonexistence of the graphs with diameter 2 and defect 2 for any degree $d > 5$.

Mixed Moore graphs generalize both undirected and directed Moore graphs. In [26], Nguyen, Miller and Gimbert proved that all known mixed Moore graphs of diameter 2 are unique and that mixed Moore graphs of diameter $k \geq 3$ do not exist.

2.2. Radial Moore graphs

Radial Moore graphs appear into the context of Moore graphs. Since the diameter of a Moore graph is equal to its radius, one can consider graphs in which the condition on the diameter is relaxed just by one, while the condition on the radius is maintained. More precisely, radial Moore graphs are regular graphs of degree d, radius k, diameter $\leq k+1$ and order the Moore bound $M_G(d,k)$.

There are exactly five radial Moore graphs of degree $d=3$ and radius $k=2$ (one of them is the unique cubic Moore graph: the Petersen graph). Capdevila, Conde, Exoo, Gimbert and López in [3] proposed different classifications of radial Moore graphs according to their proximity to extremal properties of a Moore graph. They also enumerate all radial Moore graphs for the cases $(d,k) \in \{(3,2),(3,3),(4,2)\}$ and ranked them. In particular, the 'closest' radial Moore graph to a Moore graph for each one of that (d,k) cases was found (see Figure 4).



(3, 2) (3, 3) (4, 2)

Figure 4: Several radial Moore graphs for different values of (d,k).

It is not difficult to find radial Moore graphs of radius $k=2$ (diameter 3) and any degree $d \geq 3$. However, the problem for $k \geq 3$ is much more complicated. Exoo, Gimbert, López and Gómez in [9] proved the existence of radial Moore graphs of radius three. They also settled the existence problem for (d,k) equal to $(3,4)$, $(4,4)$, $(5,4)$, and $(3,5)$. It is conjectured that, except for some small values of k, radial Moore graphs do not exist, but a proof is not yet in sight.

Regarding the directed case, Knor [23] proved that these extremal digraphs do exist for any value of $d \geq 1$ and $k \geq 1$. In [19], Gimbert and López introduced a digraph operator, based on the line digraph, which allowed to construct new radially Moore digraphs and recover the known ones.

2.3. Multipartite digraphs

Some other aspects in which Joan Gimbert contributed deal with the study of bipartite digraphs. It is known [13] that order N of a strongly connected bipartite digraph in terms of its diameter k and the maximum out-degrees (d_1, d_2) of its partite sets of vertices is bounded by the *Moore-like bound*

$$M(d_1, d_2, k) = \begin{cases} 2(1 + d_1d_2 + \cdots + (d_1d_2)^m) & \text{if } k = 2m+1, \\ (d_1 + d_2)(1 + d_1d_2 + \cdots + (d_1d_2)^{m-1}) & \text{if } k = 2m. \end{cases}$$

Figure 5 shows how to derive the Moore-like bound for even and odd diameter.

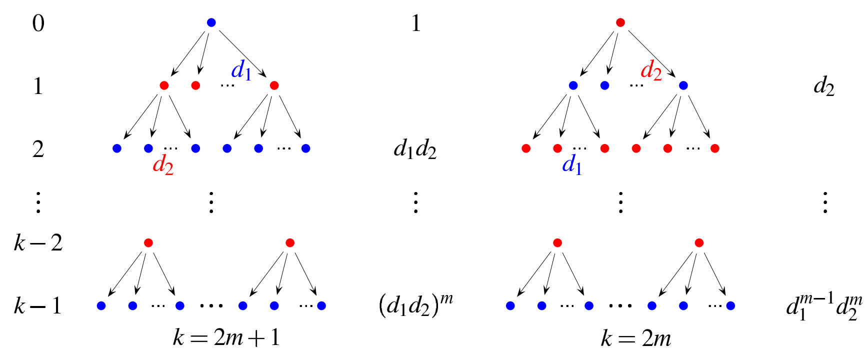


Figure 5: Local structure of a bipartite digraph.

It has been proved that, when $d_1d_2 > 1$, the digraphs attaining such a bound only exist when $2 \le k \le 4$. By using the theory of circulant matrices and the so-called De Bruijn near–factorizations of cyclic groups, Fiol, Gimbert, Gómez and Wu in [11] presented some new constructions of Moore bipartite digraphs of diameter $k = 3$ and composite out–degrees. Moreover, it is shown that any Moore bipartite digraph of diameter 4 is the line digraph of a Moore bipartite digraph of diameter 3. In a previous work [24], Gimbert and Wu gave conditions for some particular $(0, 1)$-matrix equations to ensure that the $(0, 1)$-matrix solutions have an underlying line digraph structure.

In [10], Fiol and Gimbert consider almost Moore bipartite digraphs, that is, bipartite digraphs which miss the Moore-like bound by just one vertex in each partite set. They give some necessary conditions for the existence of such digraphs in the case of odd diameter. As a consequence, it is shown that any almost Moore bipartite digraph of diameter 5 is the second order line digraph of a Moore bipartite digraph of diameter 3.

The study of bipartite digraphs can be extended to multipartite digraphs, assuming that every vertex of a given partite set is adjacent to the same number of vertices in each of the other independent sets. Comellas, Fiol, Gimbert and Mitjana in [4, 5, 6] introduced the concept of *weakly distance–regular* and studied some of its main properties.

In [12], Fiol, Gimbert and Miller determined when a Moore multipartite digraph is weakly distance–regular. Within this framework, some necessary conditions for the existence of a Moore r-partite digraph with interpartite out-degree $\delta > 1$ and diameter $k = 2m$ are obtained. In the case $\delta = 1$, which corresponds to almost Moore digraphs, a necessary condition in terms of the permutation cycle structure is derived. Also, some constructions of dense multipartite digraphs of diameter two that are vertex-transitive were presented.

2.4. Eccentric digraphs

Joan Gimbert's contributions were also related to eccentricities in graphs and digraphs. The eccentricity of a vertex u is the maximum distance from u to the rest of the vertices of the graph. The computation of the eccentricity of all of the vertices of a graph, that is, its *eccentricity sequence*, is 'easy'. Nevertheless, the converse problem, which consists in determining if given a sequence of positive integers there is a graph with such an eccentricity sequence, becomes difficult.

The characterization of some integer sequences as eccentricity sequences of digraphs were studied in [18] in Nacho López's PhD thesis supervised by Joan Gimbert. They found necessary conditions for a positive integer sequence $S : s_1, s_2, \ldots, s_n$ to be an eccentricity sequence of a digraph and characterize the eccentricity behaviour of those sequences S such that $\max(S) = n-1$ and $\min(S) \geq n-3$. As an application, they proved that the sequence

$$S : 4, 4, 5, 6, 6, 6, 6$$

is the first shortest sequence, given in lexicographic order, satisfying the necessary conditions which is not the eccentricity sequence of any digraph.

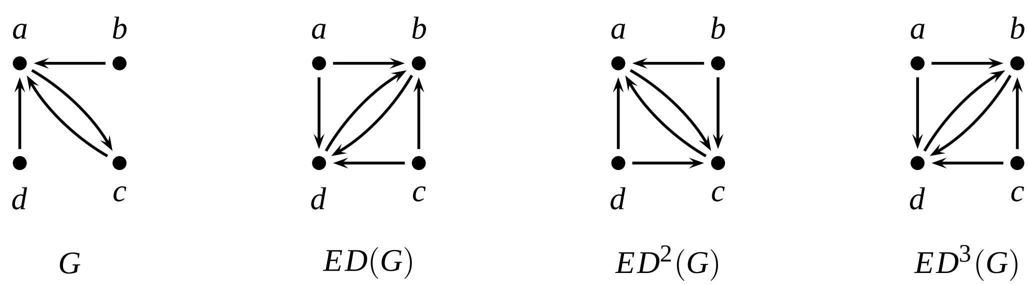


Figure 6: The iterated sequence of eccentric digraphs for a digraph G.

Joan Gimbert was interested in another related problem regarding eccentricities in digraphs: The *eccentric digraph* $ED(G)$ of a [di]graph G has the same vertex set as G but with an arc from a vertex u to a vertex v if and only if v is a farthest vertex from u in G, that is, v is an eccentric vertex of u. Again, it is 'easy' to compute the eccentric digraph of a [di]graph, but the converse problem is much more difficult. Gimbert, López, Miller and Ryan in [20] gave a characterization of those [di]graphs which are an eccentric digraph of some [di]graphs. Given a positive integer k, the k^{th} iterated eccentric digraph of G is defined as

$$ED^k(G) = ED(ED^{k-1}(G))$$

where $ED^0(G) = G$. Another of Joan Gimbert's contributions together with Miller, Ruskey and Ryan in [22] was concerning the iterated sequence of eccentric digraphs (see Figure 6)

$$G, ED(G), ED^2(G), \ldots .$$

In particular, Gimbert, López, Miller and Ryan in [21] show that, for 'almost' every digraph G, this sequence has period two and tail zero, that is,

$$G = ED^2(G) = ED^4(G) = \cdots \quad \text{and} \quad G \neq ED(G) = ED^3(G) = \cdots .$$

On the other hand, in [27], Villaró, Gimbert and Béjar presented an algorithm that, given a sequence and just by doing arithmetic operations, provides a *stratified graph* with such an eccentricity sequence.

3. Joan, the colleague and friend

Joan's lifestyle goes beyond his resarch. Respectful and friendly to everyone, he was always willing to listen and help. As a teacher, Joan was much appreciated by his students. He knew how to encourage them; his classes were didactic, interesting and filled with useful suggestions; transmitting the great enthusiasm he felt for mathematics and for all he explained.

Joan had many other interests. The promotion of mathematics: he published several articles in informative journals about prime numbers and cryptography [16] or about Google pagerank [17]. His concern and dedication to social and environmental problems in which he was involved, and his passion for nature which he made us share with him. He was always ready to organize a weekend excursion to the mountains or a bike ride on the paths of the surrounding countryside... His departure, which has come too soon, leaves a huge void in all of us.

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