Some useful families of polynomials in the theory of graph spectra.

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Outlook

Abstract

As it is well known, from the (adjacency or Laplacian) spectrum of a graph we can infer some properties about its combinatorial structure. Some examples are the diameter of the graph, its independence number, some distance-regularity properties, etc. In order to derive such results, different families of polynomials, obtained from the spectrum, have shown to be very useful. In this talk, we aim to present some of these families, together with their applications.

Preliminaries

Graphs and spectra

Let $G = (V, E)$ be a graph with $n = |V|$ vertices, $m = |E|$ edges, and adjacency matrix $\boldsymbol{A}$ with spectrum

$$\operatorname{sp} G = \{\theta_0, \theta_1^{m_1}, \cdots, \theta_d^{m_d}\}.$$

where $\theta_0 > \theta_1 > \cdots > \theta_d$.
When the eigenvalues are presented with possible repetitions, we shall indicate them by

$$\operatorname{ev} G : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Some combinatorial graph properties

Given a graph G...

- The *diameter* D is the maximum distance between vertices
- The *k-independence* of G, $\alpha_k = \alpha_k(G)$, is the size of the largest set of vertices such that any two vertices in the set are at distance larger than k (in particular, $\alpha_1 = \alpha$ is the independence number of a graph)
- G is distance-regular if the number of ℓ-walks between two vertices only depend of the distance between them.
  Alternatively, G, with diameter d and distance matrices $A_0(= I), A_1(= A), \ldots, A_d$, is distance-regular if and only if there exists sequence of (orthogonal) polynomials $p_0, p_1, \ldots, p_d$ such that

$$A_i = p_i(A), \qquad i = 0, 1, \ldots, d.$$

- The *resistance* between two vertices is the electric resistance between them assuming that all the edges are unit resistors.

⋮

The alternating polynomials

The alternating polynomials

Let G be a (connected) graph with distinct eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. The *$k$-alternating polynomial* $P_k(x)$ is the polynomial $p \in \mathbb{R}_k(x)$ satisfying

- $|p(\theta_i)| \leq 1$ for all $i = 1, ..., d$.
- $p(\theta_0)$ is maximum.

Computing the alternating polynomials

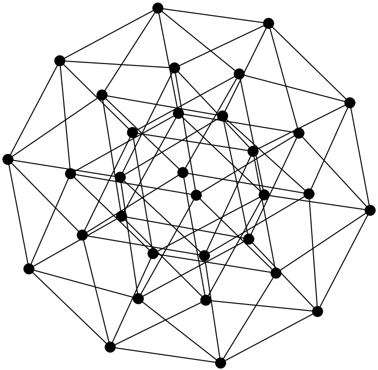
The k-alternating polynomial $P_k \in \boldsymbol{R}_k[x]$ is the polynomial defined by $P_k(\theta_i) = x_i$, $1 \leq i \leq d$, where the vector $(x_1, x_2, \ldots, x_d)$ is a solution of the following linear programming problem:

$$\begin{array}{rl} \texttt{maximize} & x_0 \\ \texttt{with constraints} & f[\theta_0, \ldots, \theta_m] = 0, \ m = k+1, \ldots, d \\ & x_i \leq 1, \ x_i \geq -1 i = 1, \ldots, d, \end{array}$$

where $f[\theta_0, \ldots, \theta_i]$ is the m-th divided differences of Newton interpolation, $f[\theta_0, \ldots, \theta_i] = \frac{f[\theta_1, \ldots, \theta_i] - f[\theta_0, \ldots, \theta_{i-1}]}{\theta_i - \theta_0}$, with $f[\theta_i] = P_k(\theta_i) = x_i$, $0 \leq i \leq d$.

An example with $d = 5$: The hypercube Q_5

$G = Q_5$ has diameter $d = 5$,
and $d + 1 = 6$ distinct eigenvalues $\operatorname{ev} Q_5 = \{-5, -3, -1, 1, 3, 5\}$.



An example with $d = 5$

Let us take the following points

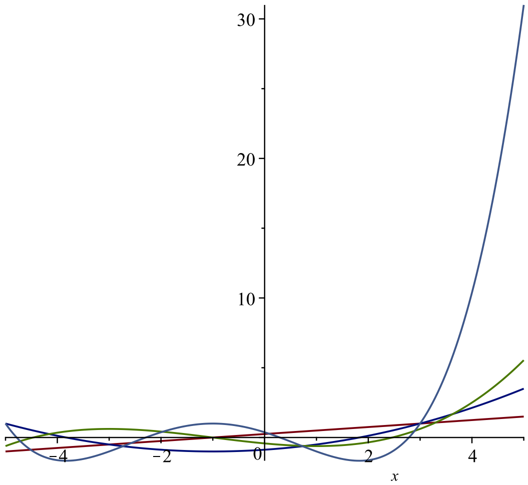
$$\theta_0 = 5,\ \theta_1 = 3,\ \theta_2 = 1,\ \theta_3 = -1,\ \theta_4 = -3,\ \theta_5 = -5.$$

Then, the corresponding k-alternating polynomials and their values at the mesh points $(\theta_5, \theta_4, ..., \theta_1)$, and θ_0 are:

$$\begin{aligned}
P_4(x) &= \frac{1}{24}(x^4 + 4x^3 - 10x^2 - 28x + 9),\ (1, -1, 1, -1, 1),\ 31;\\
P_3(x) &= \frac{1}{16}(x^3 + 3x^2 - 9x - 11),\ (-1, 1, 0, -1, 1),\ 9;\\
P_2(x) &= \frac{1}{8}(x^2 + 2x - 7),\ (1, -\frac{1}{2}, -1, -\frac{1}{2}, 1),\ 72;\\
P_1(x) &= \frac{1}{4}(x + 1),\ (-1, -\frac{1}{2}, 0, \frac{1}{2}, 1),\ 32.
\end{aligned}$$

Note that, in this example, P_k takes $k + 1$ alternating values ± 1 at the mesh points other than θ_0, as does the Chebychev polynomial T_k in $[-1, +1]$.

The alternating polynomials of Q_5



The diameter D vs. P_k

The polynomial $P_k(x)$ is known to be unique, and it was first used to study the relationship between the spectrum of a graph and its diameter.

Theorem (F., Garriga, Yebra, 1996)

Let G be a graph on n vertices, with distinct eigenvalues $\theta_0 > \cdots > \theta_d$. Let $\boldsymbol{\nu}$ be the (positive) θ_0-eigenvector with minimum component 1. Let $P_k(x)$ be the k-alternating polynomial. Then,

$$P_k(\theta_0) > \|\boldsymbol{\nu}\|^2 - 1 \quad \Rightarrow \quad D(G) \le k.$$

The k-independece number α_k vs. P_k

Theorem (F., 1997)

Let G be a d-regular graph on n vertices, with distinct eigenvalues $\theta_0 > \cdots > \theta_d$ and let $P_k(x)$ be its k-alternating polynomial. Then,

$$\alpha_k \leq \frac{2n}{P_k(\theta_0)+1}.$$

Moreover, G is an r-antipodal distance-regular graph if and only if its distance-d graph G_d is constituted by disjoint copies of the complete graph K_r with

$$r = \alpha_d = \frac{2n}{P_k(\theta_0)+1} = 2n\left(\sum_{i=0}^{d}\frac{\pi_0}{\pi_i}\right)^{-1}.$$

The predistance polynomials

The predistance polynomials

Given a graph G with spectrum as above, the *predistance polynomials* $p_0, \ldots, p_d$, (F. and Garriga, 1997), are the orthogonal polynomials with respect to the scalar product

$$\langle f, g \rangle_G = \frac{1}{n} \operatorname{tr}(f(\boldsymbol{A}) g(\boldsymbol{A})) = \frac{1}{n} \sum_{i=0}^{d} m_i f(\lambda_i) g(\lambda_i),$$

normalized in such a way that $\|p_i\|_G^2 = p_i(\lambda_0)$ (we know that $p_i(\lambda_0) > 0$ for every $i = 0, \ldots, d$).

Properties

...with properties:

1. Orthogonal sequence

$$\operatorname{dgr} p_i = i, (i = 0, \ldots, d), \qquad p_i \perp p_j, (i \neq j).$$

2. Interlacing of zeros
3. Three-term recurrence and specific values
   $xp_i = \beta_{i-1}p_{i-1} + \alpha_i p_i + \gamma_{i+1}p_{i+1}, \quad (\beta_{i-1} = \gamma_{d+1} = 0)$,
4. Preintersection numbers
5. Sum polynomials
6. Recovering the spectrum
7. ...

Distance polynomials

If G is distance-regular, the predistance-polynomials become, respectively, the distance polynomials $p_0, p_1, \ldots, p_d$ giving the distance matrices:

$$p_i(\boldsymbol{A}) = \boldsymbol{A}_i \qquad i = 0, 1, \ldots, d;$$

and the number of vertices at distance i from every vertex u is:

$$k_i = |G_i(u)| = p_i(\lambda_0), \qquad i = 0, \ldots, d(= D).$$

The spectral excess theorem (SPET)

Theorem (F., Garriga, 1997)

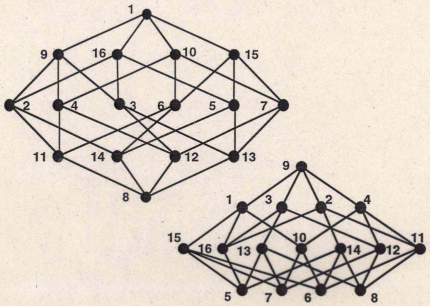
A connected regular graph with $d+1$ distinct eigenvalues is distance-regular if and only if its average excess equals its spectral excess:

$$\overline{k}_d = p_d(\lambda_0) = n\left(\sum_{i=0}^{d} \frac{\pi_0^2}{m(\lambda_i)\pi_i^2}\right)^{-1},$$

where $\pi_i = \prod_{j\neq i} |\lambda_i - \lambda_j|$ *for* $i = 0, 1, \ldots, d$*.*

The Hoffman graph

The Hoffman graph



- The Hoffman graph is cospectral with the hypercube Q_4, but it is **not** distance-regular

A generalization of the SPET

Theorem (Dalfó, Van Dam, F., Garriga, Gorissen, 2011)

Let G be a regular graph with n vertices, diameter D, and $d+1$ distinct eigenvalues. For some integer $j \le D$, let $q_j = p_0 + \cdots + p_j$, $\boldsymbol{S}_j = \boldsymbol{A}_0 + \cdots + \boldsymbol{A}_j$, and $\overline{s}_j = \|\boldsymbol{S}_j\|^2 = \frac{1}{n}\sum_{u\in V} s_j(u)$. Then,

$$q_j(\lambda_0) \le \overline{s}_j,$$

and the equality holds if and only if

$$q_j(\boldsymbol{A}) = \boldsymbol{S}_j.$$

Moreover, G is m-partially distance-regular if and only if equality holds for $j = m-1, m$.

The minor polynomials

The minor polynomials

Let $G = (V, E)$ be a graph with spectrum $\operatorname{sp} G = \{\theta_0 > \theta_1^{m_1} > \cdots > \theta_d^{m_d}\}$.
The *k-minor polynomial* $p_k \in \boldsymbol{R}_k[x]$ is the polynomial defined by $p_k(\theta_0) = 1$ and $p_k(\theta_i) = x_i$, $1 \le i \le d$, where the vector $(x_1, x_2, \ldots, x_d)$ is a solution of the following linear programming problem:

$$\begin{array}{rl} \texttt{minimize} & \sum_{i=0}^{d} m_i p(\theta_i) \\ \texttt{with constraints} & f[\theta_0, \ldots, \theta_m] = 0,\ m = k+1, \ldots, d \\ & x_i \ge 0,\ i = 1, \ldots, d, \end{array}$$

where $f[\theta_0, \ldots, \theta_i]$ is the m-th divided differences of Newton interpolation, $f[\theta_0, \ldots, \theta_i] = \frac{f[\theta_1, \ldots, \theta_i] - f[\theta_0, \ldots, \theta_{i-1}]}{\theta_i - \theta_0}$, with $f[\theta_i] = p_k(\theta_i) = x_i$, $0 \le i \le d$.

The Hamming graph $H(2,7)$

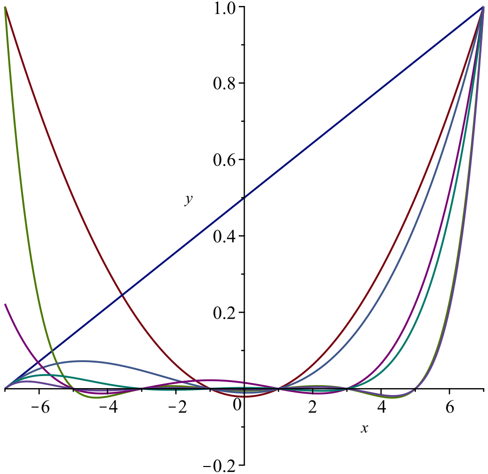
The Hamming graph $H(2,7)$ or $7-cube$ is a distance-regular graph with $n=2^7$ vertices, degree $k=7$, diameter $D=7$, and spectrum

$$7^1,\ 5^7,\ 3^{21},\ 1^{35},\ -1^{35},\ -3^{21},\ -5^7,\ -7^1$$

k	x_7	x_6	x_5	x_4	x_3	x_2	x_1	x_0
1	0	1/7	2/7	3/7	4/7	5/7	6/7	1
2	1	1/2	1/6	0	0	1/6	1/2	1
3	0	1/14	1/21	0	0	5/42	3/7	1
4	2/9	0	0	1/45	0	0	2/9	1
5	0	1/35	0	0	0	0	6/35	1
6	1	0	0	0	0	0	0	1
7	0	0	0	0	0	0	0	1

Table: Values $x_i = p_k(\theta_i)$ of the k-minor polynomials of the Hamming graph $H(2,7)$.

The minor polynomials of the Hamming graph $H(2,7)$



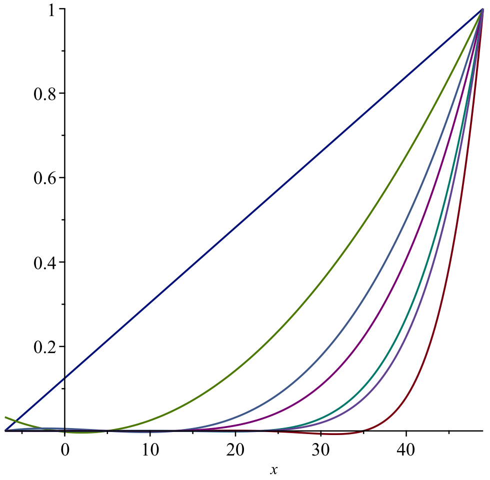
The Johnson graph $J(14, 7)$

The Johnson graph $J(14, 7)$ is a distance regular graph with $n = \binom{14}{7}$ vertices, degree $k = 49$, diameter $D = 7$, and spectrum:

i	0	1	2	3	4	5	6	7
θ_i	49	35	23	13	5	-1	-5	-7
m_i	1	13	77	273	637	1001	1001	429

Table: Eigenvalues and multiplicities of the Johnson graph $J(14, 7)$.

The minor polynomials of the Johnson graph graph $J(14, 7)$



The k-independence number vs p_k

Theorem

Let G be a k-partially walk-regular graph with n vertices, adjacency matrix $\boldsymbol{A}$, and spectrum

$$\operatorname{sp} G = \{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}.$$

Let $p_k \in \mathbb{R}_k[x]$ be the k-minor polynomial. Then,

$$\alpha_k \le \operatorname{tr} p_k(\boldsymbol{A}) = \sum_{i=0}^{d} m_i p_k(\theta_i). \tag{1}$$

k	3	4	5	6	7
$P_k(\theta_0)$	1115/81	485/9	8629/25	3431	–
$q_k(\theta_0)$	1716	2941	3382	3431	3432
Bound from Theorem 2	464	125	20	2	–
Bound from Theorem 5	19	6	2	2	1

Table: Comparison of the bounds for α_k in the Johnson graph $J(14, 7)$.

k	1	2	3	4	5	6	7
Bound from Theorem 5	64	16	8	3	2	2	1

Table: Bounds for α_k in the Hamming graph $H(2, 7)$.

Two particular cases

- When $k = 1$, α_1 coincides with the standard independence number α. In this case the minor polynomial is $p_1(x) = \frac{x-\theta_d}{\theta_0-\theta_d}$ and we obtain

$$\alpha \leq n\frac{-\theta_d}{\theta_0 - \theta_d},$$

which is known as the Hoffman's bound.

- When $k = 2$, the minor polynomial turs out to be $p_2(x) = \frac{(x-\theta_i)(x-\theta_{i-1})}{(\theta_d-\theta_i)(\theta_d-\theta_{i-1})}$, where θ_i be the largest eigenvalue such that $\theta_i \leq -1$. Then, the 2-independence number satisfies

$$\alpha_2 \leq n\frac{\theta_0 + \theta_i\theta_{i-1}}{(\theta_0 - \theta_i)(\theta_0 - \theta_{i-1})}.$$

Open problems (for the last family)

- Closed formulas for $k > 2$.
- Study the integer bound vs. cases of equality.
- Application to the existence of perfect codes.
- Relations with other parameters (diameter, etc.).
- Relation with the alternating polynomials.

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Gràcies per l'atenció
Thanks for your attention