New Directions in Network Reliability

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Outline

1. The Reliability Polynomial
   - Properties of the reliability polynomial
2. Exact computation of the reliability polynomial
   - Reduction
   - Deletion-contraction
3. Special classes of graphs
4. Design problems
5. Reliability in directed graphs
6. Reliability with node failures
7. Diameter-constrained reliability
8. Some open problems
9. Bibliography

Stochastic networks with edge failures

- Model: undirected pseudograph (multiple edges and loops are allowed)
- Nodes are perfectly reliable
- At any given time, each edge e has a probability p_e of being operational, and a probability $q_e = 1 - p_e$ of failing (p_e is constant throughout time)
- p_e is independent from p_f, for all edges $f \neq e$
- No repair is allowed
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Assumptions about the link capacity

- We will assume that the capacity of the links is *infinite*, or alternatively, that the flow of information travelling through the links is negligible in comparison with their capacity
- If this condition fails, then it can lead to *cascading failures*
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Three reliability problems

- Two-terminal reliability:
  - Given two distinguished vertices, s and t, and given the probability p of an edge being operational, what is the probability that there exists an operational path between s and t?
- All-terminal reliability:
  - Given the probability p of an edge being operational, what is the probability that there exists an operational path between any two nodes u and v?
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Complexity of the reliability problems

- All three problems (two-terminal, all-terminal, and k-terminal reliability) are $\#\mathcal{P}$-complete
- The class $\#\mathcal{P}$ consists of the problems that can be solved by a non-deterministic counting Turing machine in polynomial time
- If a problem is $\#\mathcal{P}$-complete, then it is $\mathcal{NP}$-hard. The converse is open.

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Further definitions

- Pathset: A subset $O \subseteq E$ of edges that makes the graph operational (i.e. (V, O) is operational)
  - For two-terminal reliability, a pathset is just a path between s and t
  - For all-terminal reliability, a pathset is a spanning tree
  - For k-terminal reliability, a pathset is a Steiner tree
  - A minimal pathset is called a *minpath*
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Reliability polynomial (Moore and Shannon, 1956)

Let m be the number of edges of $G = (V, E)$. For $E' \subseteq E$, let $G' = (V, E')$

$$\mathrm{Rel}(G, p) = \sum_{E' \subseteq E,\, G' \text{ is a pathset}} p^{|E'|}(1-p)^{m-|E'|}$$

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N-form of the reliability polynomial

- Let m be the number of edges, and N_i be the number of pathsets with i edges
- The probability of obtaining a set of i edges is $p^i(1-p)^{m-i}$
- $\mathrm{Rel}(G,p) = \sum_{i=0}^{m} N_i p^i (1-p)^{m-i}$
- $\mathrm{Rel}(G,p)$ is a polynomial in p of degree at most m, which can be used to compare different topologies
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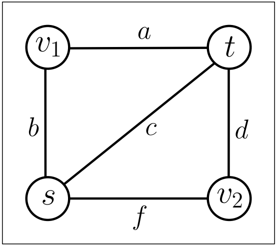
Example: Two-terminal reliability of $K_4 - \{e\}$



Pathsets:

$\{c, ab, df, ac, bc, cd, cf, abc, abd, abf, acd, acf, bcd, bcf, adf, bdf, cdf, abcd, abcf, abdf, bcdf, acdf, abcdf\}$

$N_0 = 0,\ N_1 = 1,\ N_2 = 6,\ N_3 = 10, N_4 = 5,\ N_5 = 1$

$$\begin{aligned}\mathrm{Rel}_2(G, p) &= p(1-p)^4 + 6p^2(1-p)^3 + 10p^3(1-p)^2 + 5p^4(1-p) + p^5 \\ &= p^5 - p^4 - 2p^3 + 2p^2 + p\end{aligned}$$

Alternative formulations

Using cutsets:

$$\mathrm{Rel}(G,p) = 1 - \sum_{i=0}^{m} C_i (1-p)^i p^{m-i}$$

Using complements of pathsets:

$$\mathrm{Rel}(G,p) = \sum_{i=0}^{m} F_i (1-p)^i p^{m-i}$$

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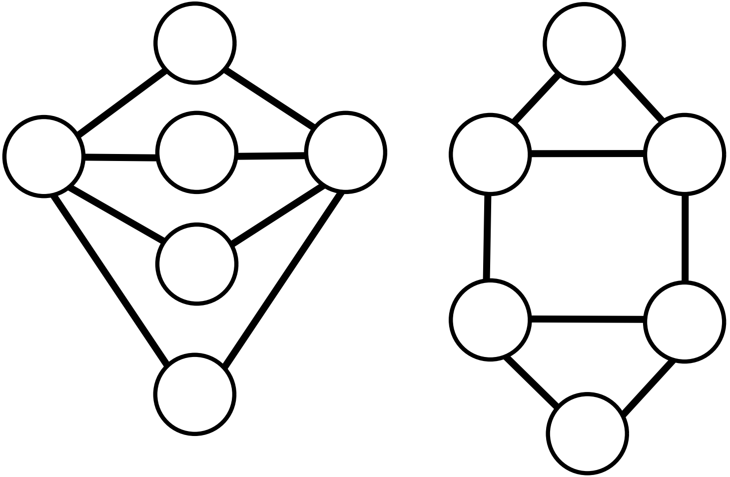
Two graphs with crossing reliability polynomials



Reliability polynomials of the two previous graphs

- $32p^5(1-p)^3 + 24p^6(1-p)^2 + 8p^7(1-p) + p^8 = -15p^8 + 56p^7 - 72p^6 + 32p^5$
- $30p^5(1-p)^3 + 25p^6(1-p)^2 + 8p^7(1-p) + p^8 = -12p^8 + 48p^7 - 65p^6 + 30p^5$
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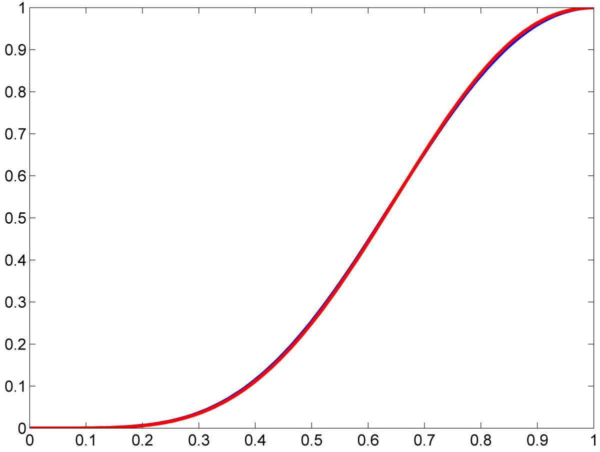
Difference between both polynomials



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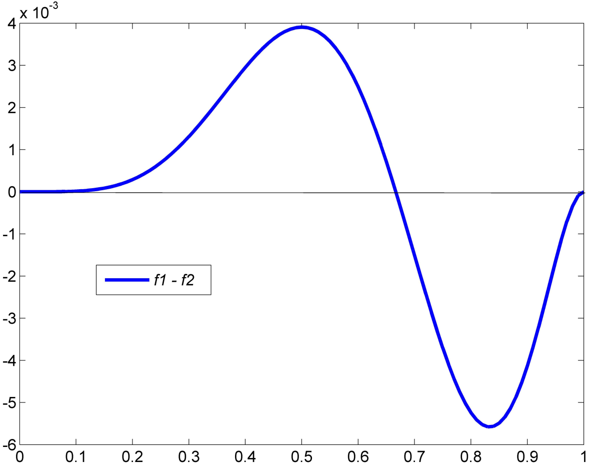

Multiplicity of the root zero

- The all-terminal reliability polynomial of G has zero as a root with multiplicity $n - 1$
- In other words, $\mathrm{Rel}_A(G, p) = p^{n-1} f(p)$, where $f(p)$ is a polynomial in p

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Some properties of the reliability polynomial

- G is a vertex-gluing of G_1 and G_2 (denoted $G_1 \cdot G_2$) if we can identify a vertex of G_1 with a vertex of G_2 in such a way as to obtain G
- If $G = G_1 \cdot G_2$, then $\mathrm{Rel}(G, p) = \mathrm{Rel}(G_1, p)\mathrm{Rel}(G_2, p)$
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Relationship with other graph parameters

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- If $k_e(G_1) = k_e(G_2) = k$, and $F_k(G_1) > F_k(G_1)$, then for p close enough to one we have $\mathrm{Rel}(G_1, p) < \mathrm{Rel}(G_2, p)$
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Summary of known coefficients

The current picture:

$$\underbrace{N_0, \ldots, N_{l-1}}_{N_i=0},\ N_l,\ \overbrace{N_{l+1}, \ldots, N_{m-c-1}}^{\text{unknown, approx.}},\ N_{m-c},\ \underbrace{N_{m-c+1}, \ldots, N_m}_{N_i=\binom{m}{m-i}}$$

Some additional coefficients may be computed efficiently

The complexity is different for the all-terminal and two-terminal cases

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Complexity of computing the exact value of different coefficients

Quantity	All-terminal	Two-terminal
l	polynomial	polynomial
c	polynomial	polynomial
$N_i, \ i < l$	polynomial	polynomial
N_l	polynomial	polynomial
N_{l+k}, with k fixed	open	polynomial
N_{m-c-k}, with k fixed	polynomial	$\#\mathcal{P}$-complete
N_{m-c}	polynomial	$\#\mathcal{P}$-complete
$N_{m-i}, \ i < c$	polynomial	polynomial
$\sum_{i=0}^{m} N_i$	$\#\mathcal{P}$-complete	$\#\mathcal{P}$-complete

Summary of known coefficients

- An F-set is a set of links whose complement is a pathset
- F_i is the number of F-sets with i edges
- $\text{Rel}(G,p) = \sum_{i=0}^{m} F_i (1-p)^i p^{m-i}$
- A set of links is either a cutset or an F-set, exclusively, hence $F_i + C_i = \binom{m}{i}$. Now set $d = m - l$:

$$\underbrace{F_0, \ldots, F_{c-1}}_{F_i = \binom{m}{i}},\ F_c,\ \overbrace{F_{c+1}, \ldots, F_{d-1}}^{\text{unknown, approx.}},\ F_d,\ \underbrace{F_{d+1}, \ldots, F_m}_{F_i = 0}$$

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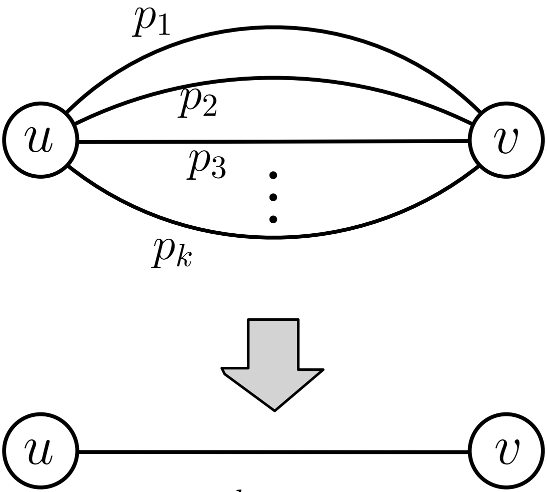
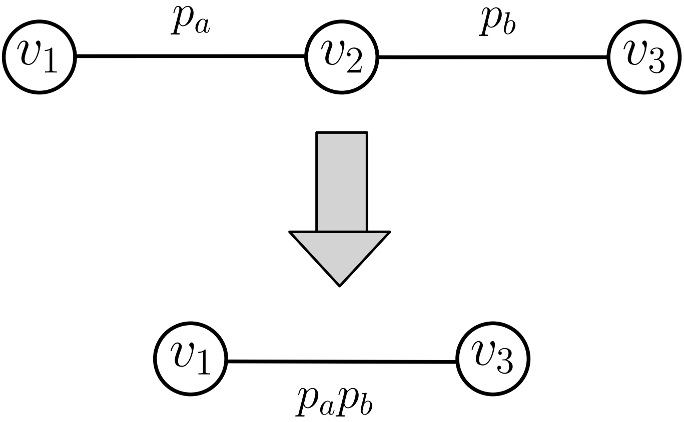
Parallel reduction

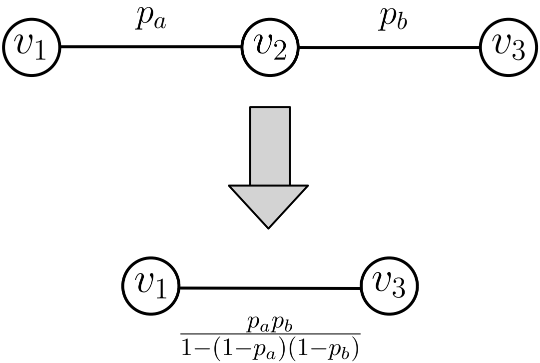


$$1 - \Pi_{i=1}^{k}(1 - p_i)$$

Series reduction



Degree-2 reduction in all-terminal and k-terminal reliability

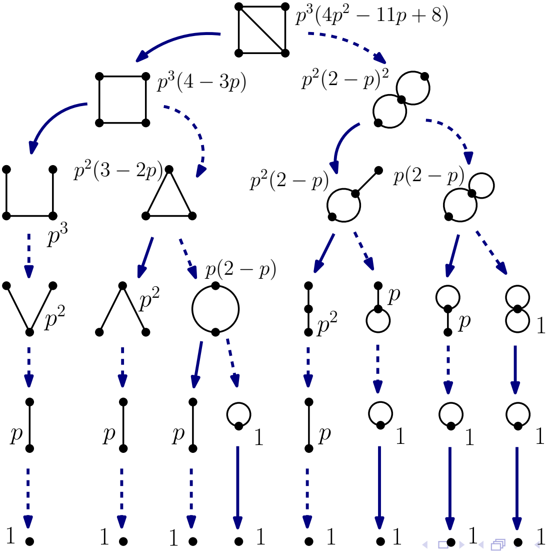


$$\mathrm{Rel}_k(G,p) = (1-(1-p_a)(1-p_b))\mathrm{Rel}_k(H,p)$$

The factoring theorem (deletion-contraction)

$$\mathrm{Rel}(G,p) = \begin{cases} 1 & \text{if } G \text{ is a singleton} \\ 0 & \text{if } G \text{ disconnected} \\ \mathrm{Rel}(G-e,p) & \text{if } e \text{ is a loop} \\ p\mathrm{Rel}(G/e,p) & \text{if } e \text{ is a cut-edge} \\ (1-p)\mathrm{Rel}(G-e,p) + p\mathrm{Rel}(G/e,p) & \text{otherwise} \end{cases}$$

All-terminal reliability polynomial of $K_4 - \{e\}$ by deletion-contraction



Number of spanning trees by deletion-contraction (Bjorklund et al.)

$$\tau(G) = \begin{cases} 1 & \text{if } G \text{ has no edges} \\ \tau(G - e) & \text{if } e \text{ is a loop} \\ \tau(G/e) & \text{if } e \text{ is a cut-edge} \\ \tau(G - e) + \tau(G/e) & \text{otherwise} \end{cases}$$

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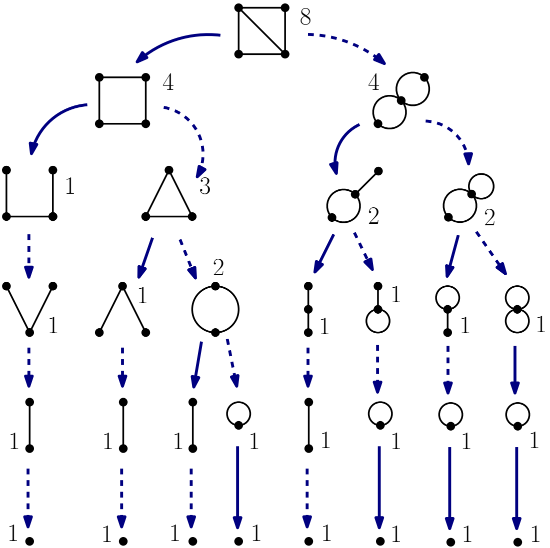
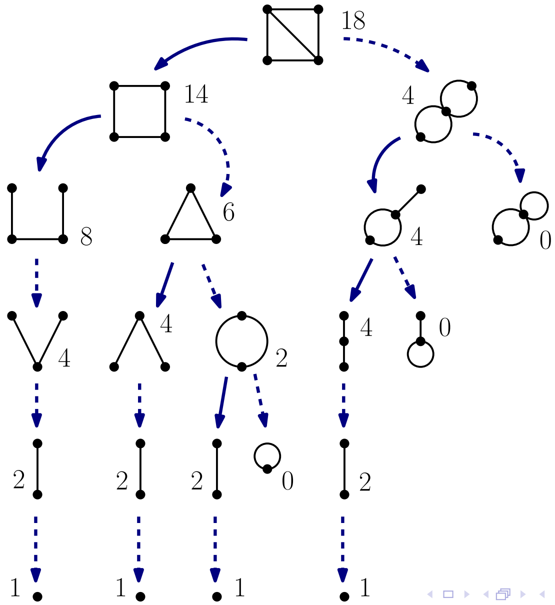


Number of acyclic orientations by deletion-contraction (Bjorklund et al.)

$$\kappa(G) = \begin{cases} 1 & \text{if } G \text{ has no edges} \\ 0 & \text{if } e \text{ is a loop} \\ 2\kappa(G/e) & \text{if } e \text{ is a cut-edge} \\ \kappa(G - e) + \kappa(G/e) & \text{otherwise} \end{cases}$$

Acyclic orientations of $K_4 - \{e\}$ by deletion-contraction



Chromatic polynomial by deletion-contraction

Denote by $P_G(t)$ the number of proper colorings of G with t colors. Let $n = n(G)$ be the number of vertices of G.

$$P_G(t) = \begin{cases} t^n & \text{if } G \text{ has no edges} \\ 0 & \text{if } e \text{ is a loop} \\ (t-1)P_{G/e}(t) & \text{if } e \text{ is a cut-edge} \\ P_{G-e}(t) + P_{G/e}(t) & \text{otherwise} \end{cases}$$

The Tutte polynomial

Notations:

- $n = n(G)$ is the number of vertices of G
- $m = m(G)$ is the number of edges of G
- $c = c(G)$ is the number of connected components of G
- For $F \subseteq E$, $c_F = c_F(G)$ denotes the number of connected components in the subgraph (V, F)

The Tutte polynomial of G:

$$T_G(x, y) = \sum_{F \subseteq E} (x-1)^{c_F - c}(y-1)^{c_F + |F| - n}$$

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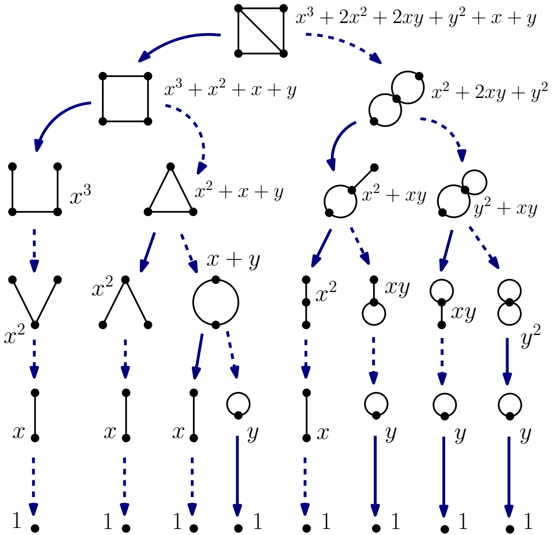
The Tutte polynomial by deletion-contraction

$$T_G(x,y) = \begin{cases} 1 & \text{if } G \text{ has no edges} \\ yT_{G-e}(x,y) & \text{if } e \text{ is a loop} \\ xT_{G/e}(x,y) & \text{if } e \text{ is a cut-edge} \\ T_{G-e}(x,y) + T_{G/e}(x,y) & \text{otherwise} \end{cases}$$

Tutte polynomial of $K_4 - \{e\}$ by deletion-contraction



Relationship between the reliability polynomial and the Tutte polynomial

The reliability polynomial can be expressed as a specialization of the Tutte polynomial:

$$\text{Rel}(G, p) = p^{n-1}(1-p)^{m-n+1} T_G(1, \frac{1}{1-p})$$

The recipe theorem

Let f be a function from graphs to the multivariate polynomial ring $\mathbb{Z}[\alpha, \beta, \gamma, \lambda, \mu]$, such that for all $e \in E$,

$$f(G) = \begin{cases} \alpha^n & \text{if } G \text{ has no edges} \\ \beta f(G - e) & \text{if } e \text{ is a loop} \\ \gamma f(G/e) & \text{if } e \text{ is a cut-edge} \\ \lambda(G - e) + \mu f(G/e) & \text{otherwise} \end{cases}$$

Then

$$f(G) = \alpha^c \lambda^{c+m-n} \mu^{n-c} T_G(\frac{\gamma}{\mu}, \frac{\beta}{\lambda})$$

The function f is called a *Tutte-Grothendieck invariant* (Welsh, 1993)

Complete graphs

- Let $A_n = \mathrm{Rel}_A(K_n)$ and $q = 1 - p$
- A_n can be computed in polynomial time with the aid of the following recursive formula

$$A_n = 1 - \sum_{j=1}^{n-1} \binom{n-1}{j-1} A_j q^{j(n-j)}$$

- Let $T_n = \mathrm{Rel}_2(K_n)$. T_n can be computed in polynomial time with the aid of the formula

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Planar graphs

- Two-terminal reliability is $\#\mathcal{P}$-complete for planar graphs
- k-terminal reliability is $\mathcal{NP}$-hard for planar graphs
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Series-parallel graphs

- A graph is series-parallel if it does not contain any subgraph homeomorphic to K_4
- Equivalently: If we apply series and parallel reductions to a series-parallel graph, we end up with a tree
- Series-parallel graphs form a subclass of planar graphs
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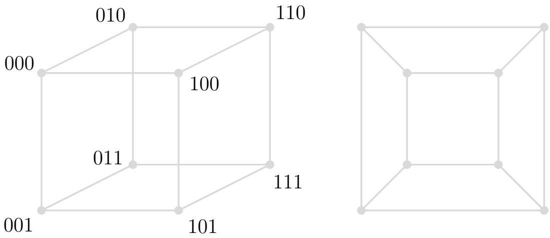
Hypercubes

- The hypercube of dimension k, or k-cube, denoted H_k, is defined as follows: The vertices are the binary strings of length k, and two vertices are joined by an edge if, and only if, they differ in only one bit (their Hamming distance is one)
- It is very popular as an architecture for supercomputers
- Two drawings of the hypercube of dimension 3:



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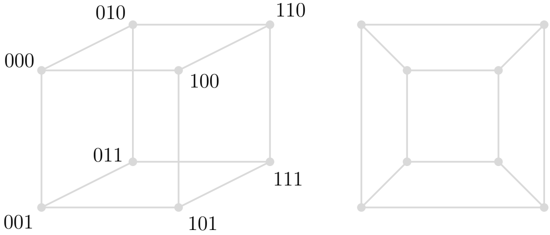

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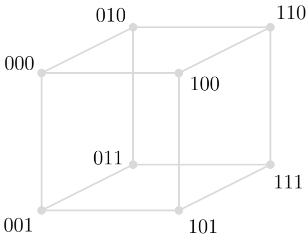

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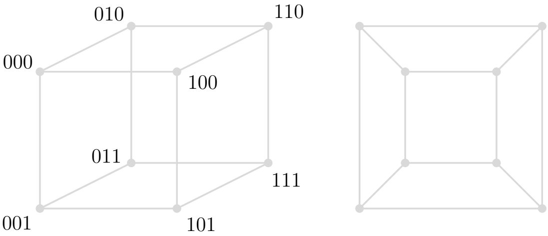

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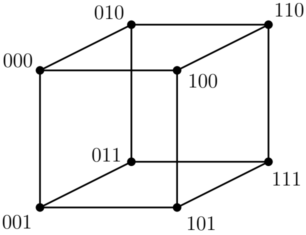

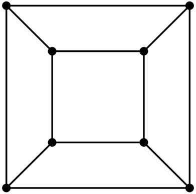

Reliability polynomial of the hypercube

- The exact reliability polynomial of H_k is not known
- There are lower and upper bounds for the reliability polynomial of H_k, obtained with the aid of the Kruskal-Katona theorem (Bulka and Dugan, 1990), and simulation (Soh and Rai, 1994)

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Cube-free graphs

- A graph is *cube-free* if it is planar and has no subgraphs homeomorphic to the 3-cube H_3
- All-terminal reliability can be solved in polynomial time for cube-free graphs (Politof and Satyanarayana, 1984)
- The algorithm uses four types of reductions:
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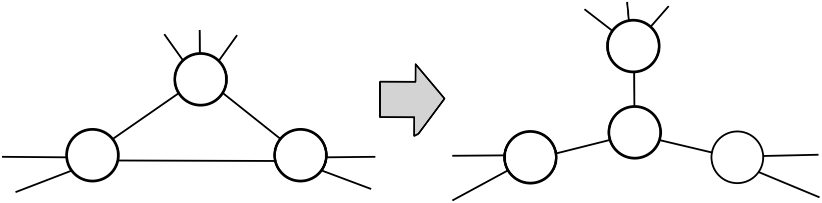

Design problem: Uniformly optimal graphs

- Uniformly optimal graph: has the highest reliability for each $0 \leq p \leq 1$
- Uniformly optimal graphs do not always exist. For example, if $m = \frac{n(n-1)}{2} - \frac{n+2}{2}$ for $n > 6$ even.
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Necessary conditions

When p is close to zero

$$\mathrm{Rel}_A(G,p) \sim N_{n-1}p^{n-1}(1-p)^{m-n+1}$$

When p is close to one

$$\mathrm{Rel}_A(G,p) \sim 1 - C_c p^{m-c}(1-p)^c$$

If G is uniformly optimal, then

- G has the highest number of spanning trees among all simple graphs with n nodes and m edges, and
- G has the highest possible edge connectivity λ among all simple graphs with n nodes and m edges (namely $\lambda = \lfloor \frac{2m}{n} \rfloor$), and the minimum number of cutsets of size λ among all such max-λ graphs

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Reliability problems in directed graphs

- s, t-connectedness:
  - Given two distinguished vertices, s and t (source and target), and given the probability p of an arc being operational, what is the probability that there exists an operational path from s to t?
- s, T-connectedness:
  - Given the probability p of an arc being operational, a source node s, and a set T of target nodes, what is the probability that there exists an operational path from s to any target node $t \in T$?
- Reachability:
  - Given the probability p of an arc being operational, and a source node s, what is the probability that there exists an operational path from s to any other node v?
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- s, t-connectedness and s, T-connectedness are $\#\mathcal{P}$-complete for acyclic digraphs
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Some design results on digraph reliability

- If we allow multiple arcs, then for every $n \geq 2$ and every $m \geq n$ there exists a uniformly optimal (m, n)-digraph (without loops)
- For all positive n and k, with $m = n + k \leq n(n+1)$, and $0 \leq k \leq 3$, there exists a uniformly optimal simple (m, n)-digraph
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Reliability polynomial for nodes

- Links are perfectly reliable, and nodes fail independently with probability $1 - p$ (i.e. they operate with probability p)
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Some uniformly optimal graphs

- The k-partite graphs $K(b, b, \dots, b, b+1, \dots, b+1)$ are uniformly optimal in the class of all graphs with $n = bh + (b+1)(k-h)$ and $m = h + 2bh + h^2 - k - 2bk - b^2k - 2hk - 2bhk + k^2 + 2bk^2 + b^2k^2$, where k is the total number of partite sets, h is the number of partite sets of size b, and $b \geq 2$, $k \geq 1$ (attributed to Bermond).
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Negative results

- The class of graphs with $m = rn + k$, $2 \leq r \leq n/5 - 4$, $0 \leq k < n/2$ does not contain a uniformly optimal graph (Goldschmidt et al., 1994).
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Diameter-constrained reliability

- $\mathrm{Rel}^D(G, p)$ represents the probability that the nodes of interest will remain connected by a path of length D or less (Petingi and Rodriguez, 2001).
- Again, there are three cases: two-terminal, all-terminal, and k-terminal
- In the two-terminal case, a pathset is an s, t-path of length $\leq D$
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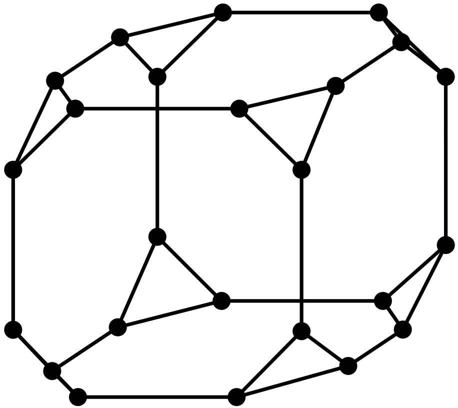
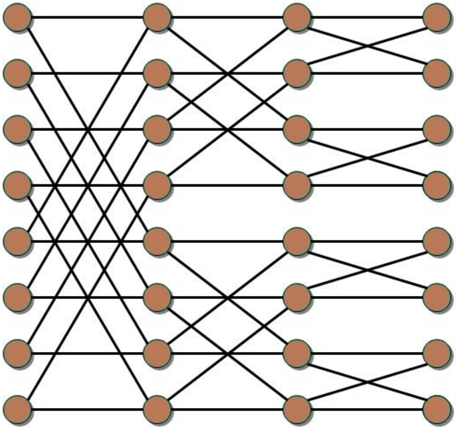
Problems I

Compute (or approximate) the reliability polynomial of the Cube-Connected-Cycles (below)



Problems II

Compute (or approximate) the reliability polynomial of the butterfly (below)



Problems III

- Compute (or approximate) the reliability polynomial of other classes of graphs
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