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An introduction to Isogeny-based cryptography

with a view towards understanding SQIsign

Benjamin Wesolowski, CNRS and ENS de Lyon

April 2025, *The SQIparty — a Workshop on Isogeny-Crypto*, Lleida, Spain

The Isogeny problem

what even is an isogeny?

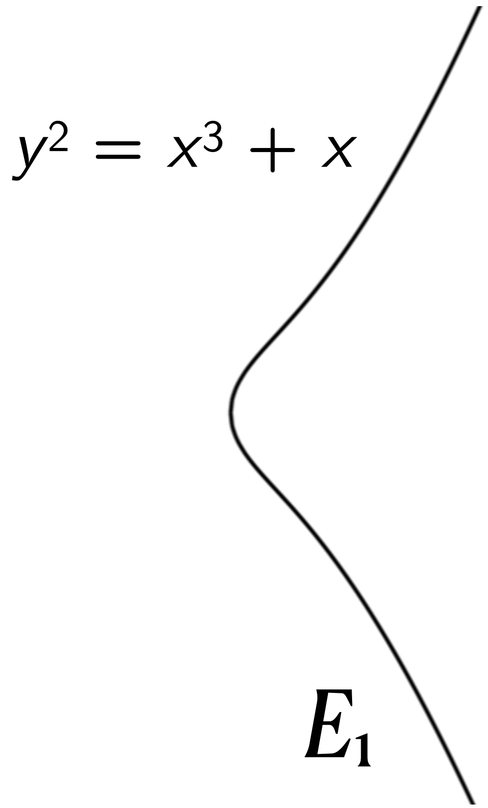
Picture by Beppe Rijs

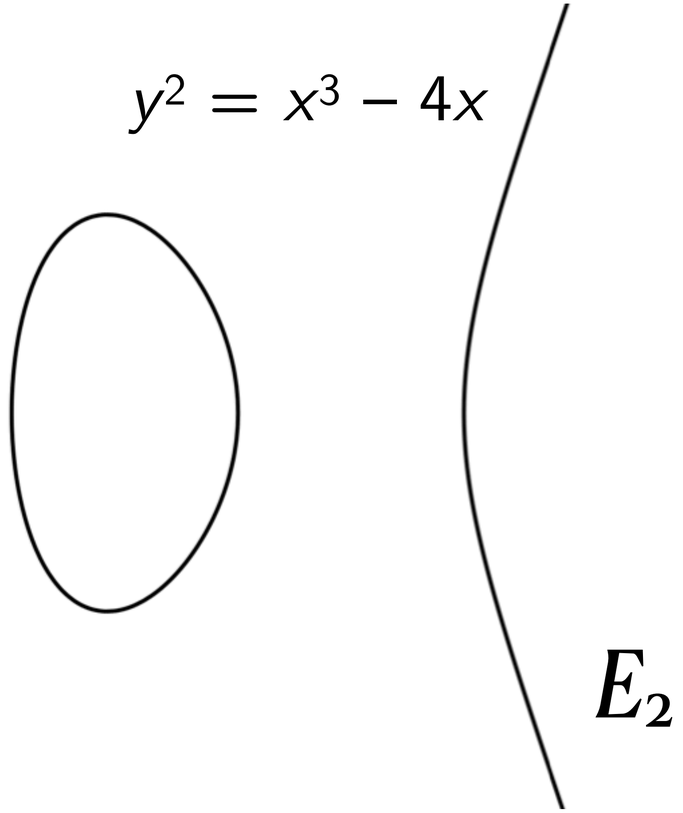
- In crypto, we use elliptic curves over a **finite field**
- Elliptic curves are **groups**: you can add points together!

Elliptic curves

equations of the form

$$y^2 = x^3 + ax + b$$





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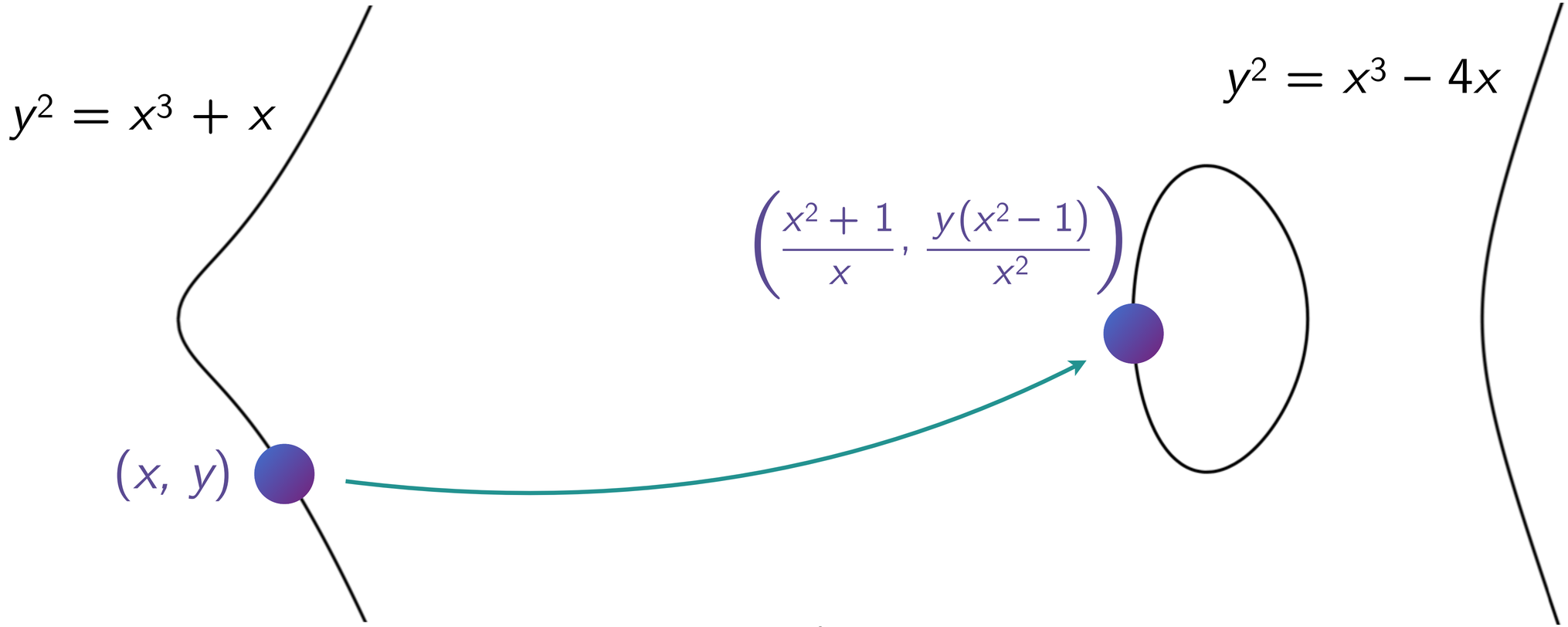
Sometimes, there is a formula to transform solutions from one equation to another



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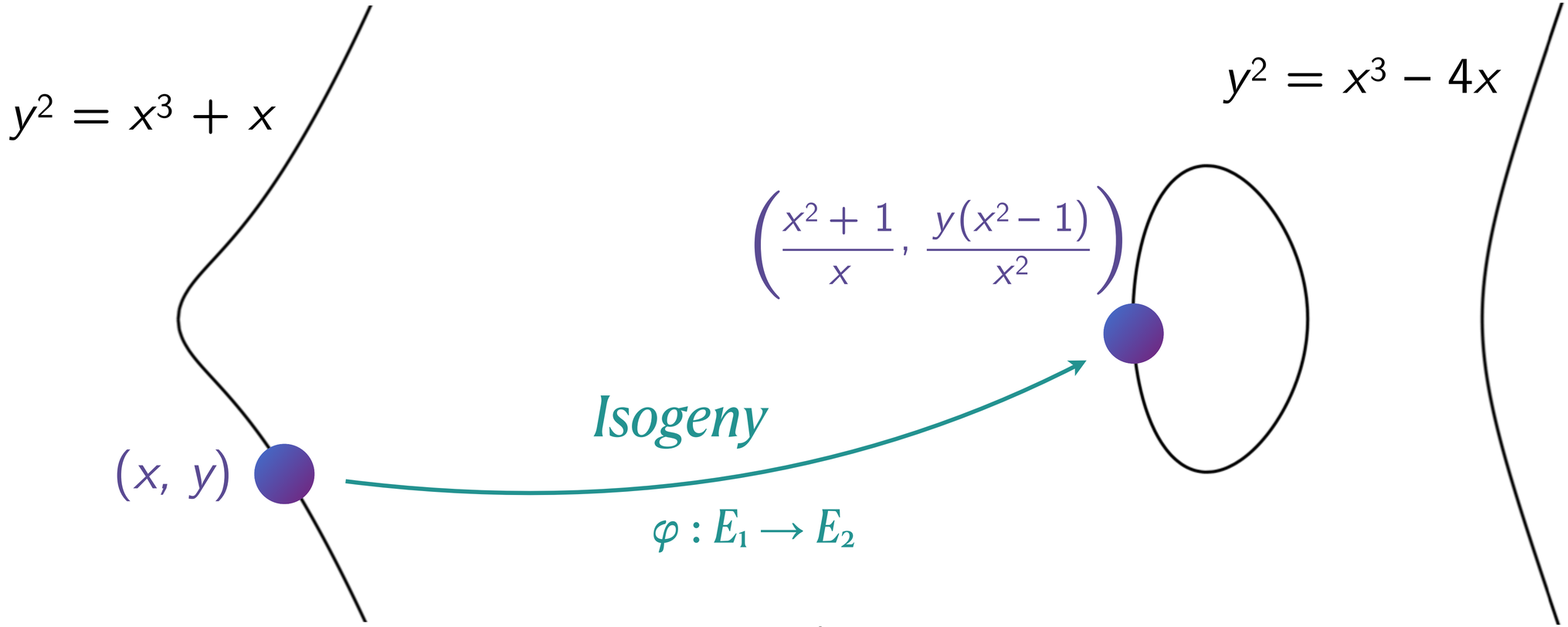

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- Isogenies are **group homomorphisms**
- **Degree** = size of kernel

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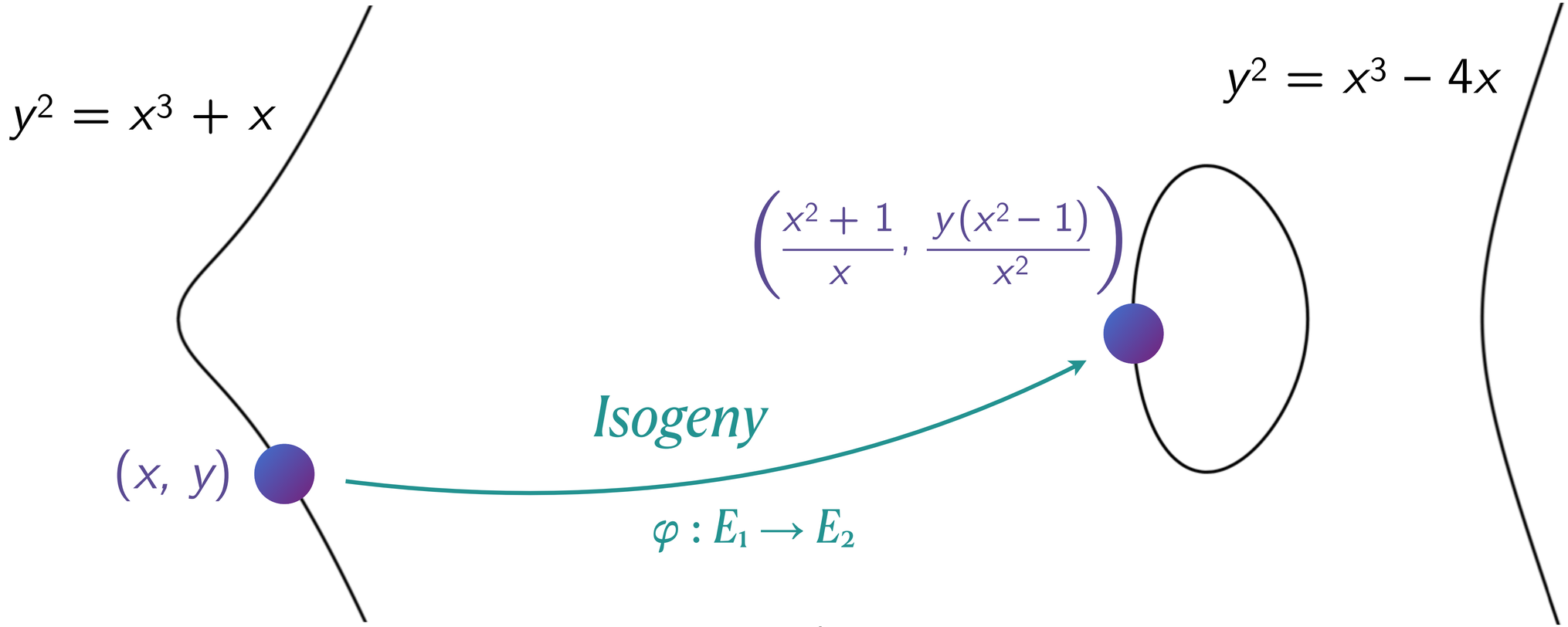

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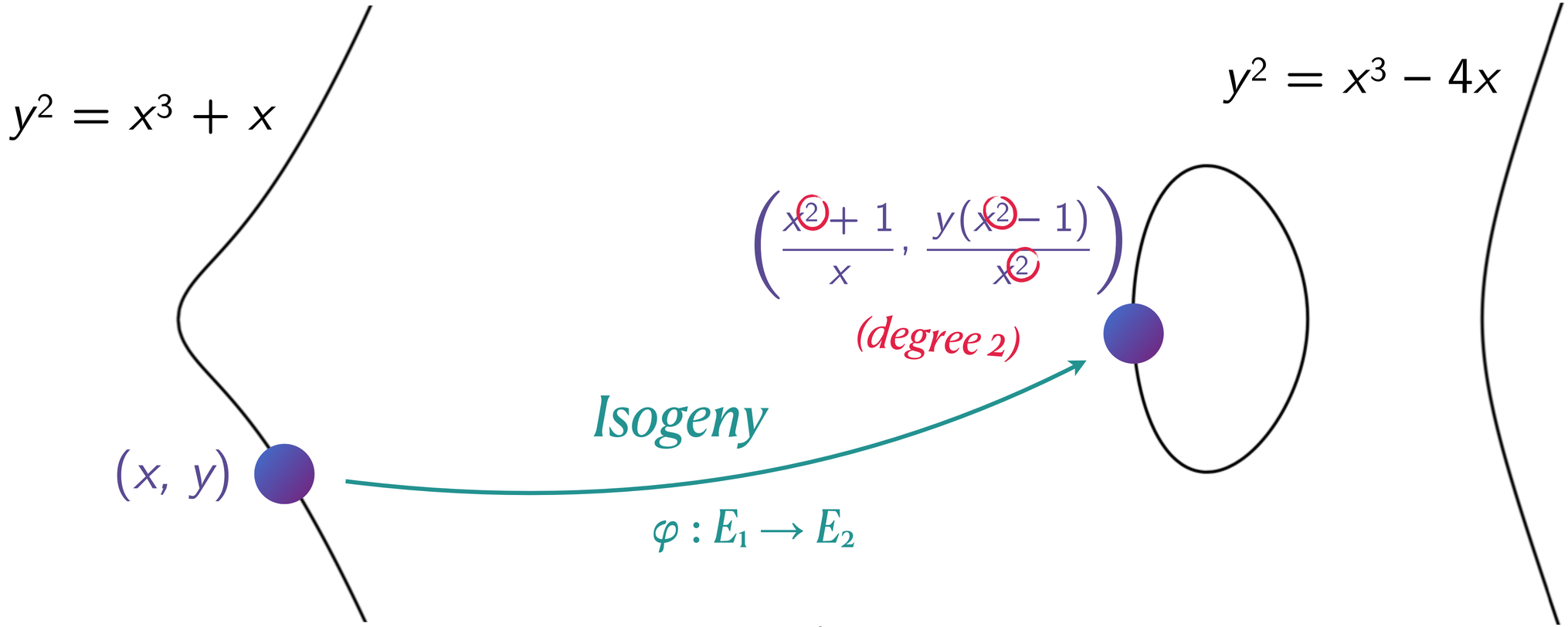

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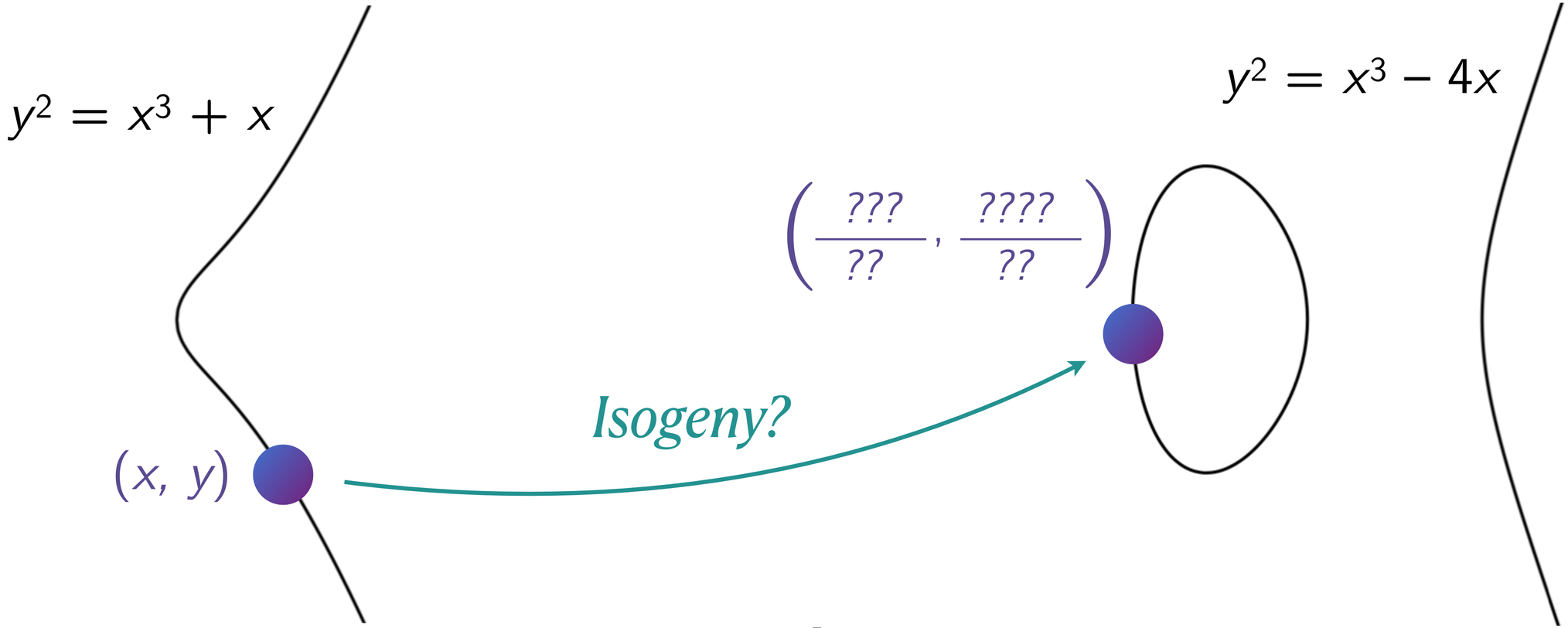
Given E_1 and E_2 find an isogeny $\varphi : E_1 \to E_2$



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$$\deg(\varphi \circ \psi) = \deg(\varphi) \cdot \deg(\psi)$$

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- Build "big" isogenies as *formal combinations* of "small" ones
  - $\varphi \circ \psi$ represented by ('comp', φ, ψ) where φ and ψ are composable
  - $\varphi + \psi$ represented by ('add', φ, ψ) where φ and ψ are both $E_1 \to E_2$

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- How to represent an isogeny?
  - any **efficient representation**: an encoding which allows one to evaluate $\varphi(P)$ in polynomial time for any P

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Isogeny graph

an isogeny of degree 2 = an edge in a graph

Isogeny graph

- **The 2-isogeny graph** (supersingular...)

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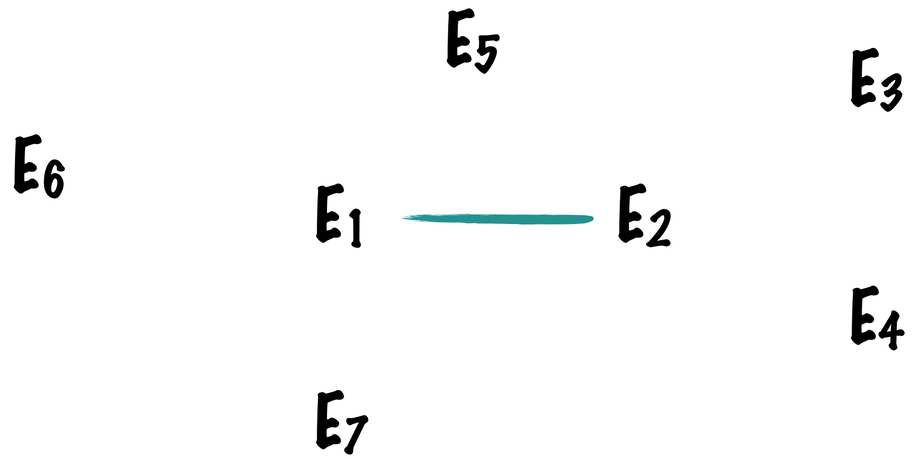

Isogeny graph

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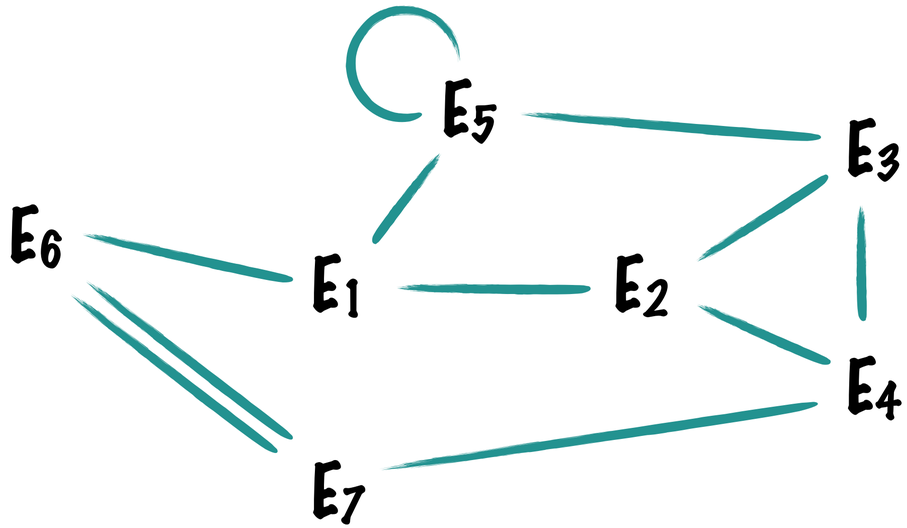

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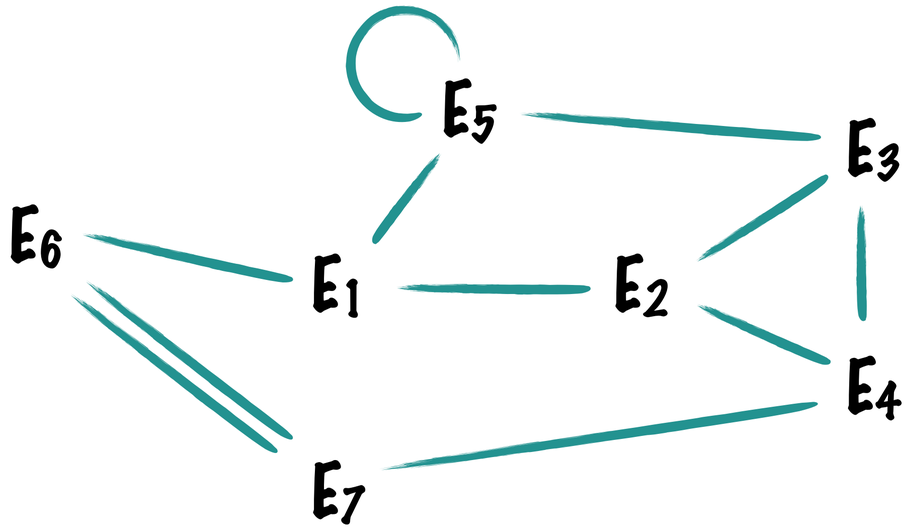

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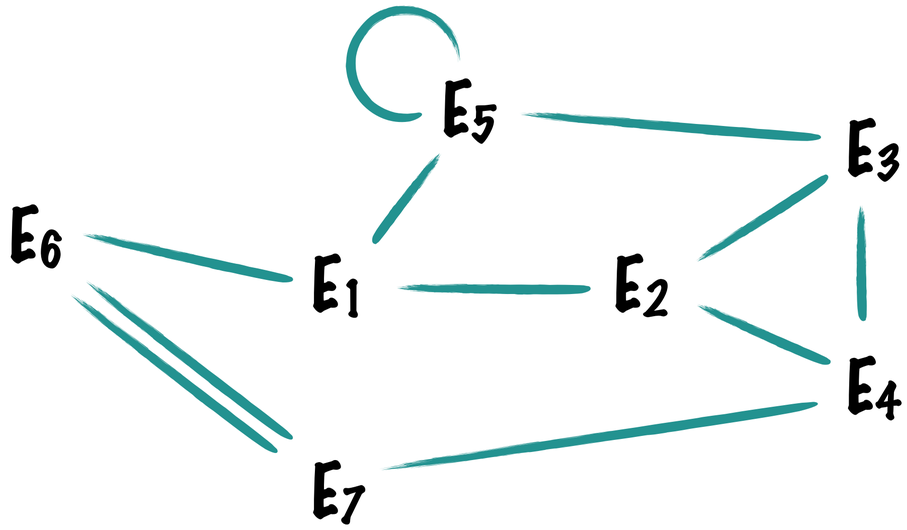

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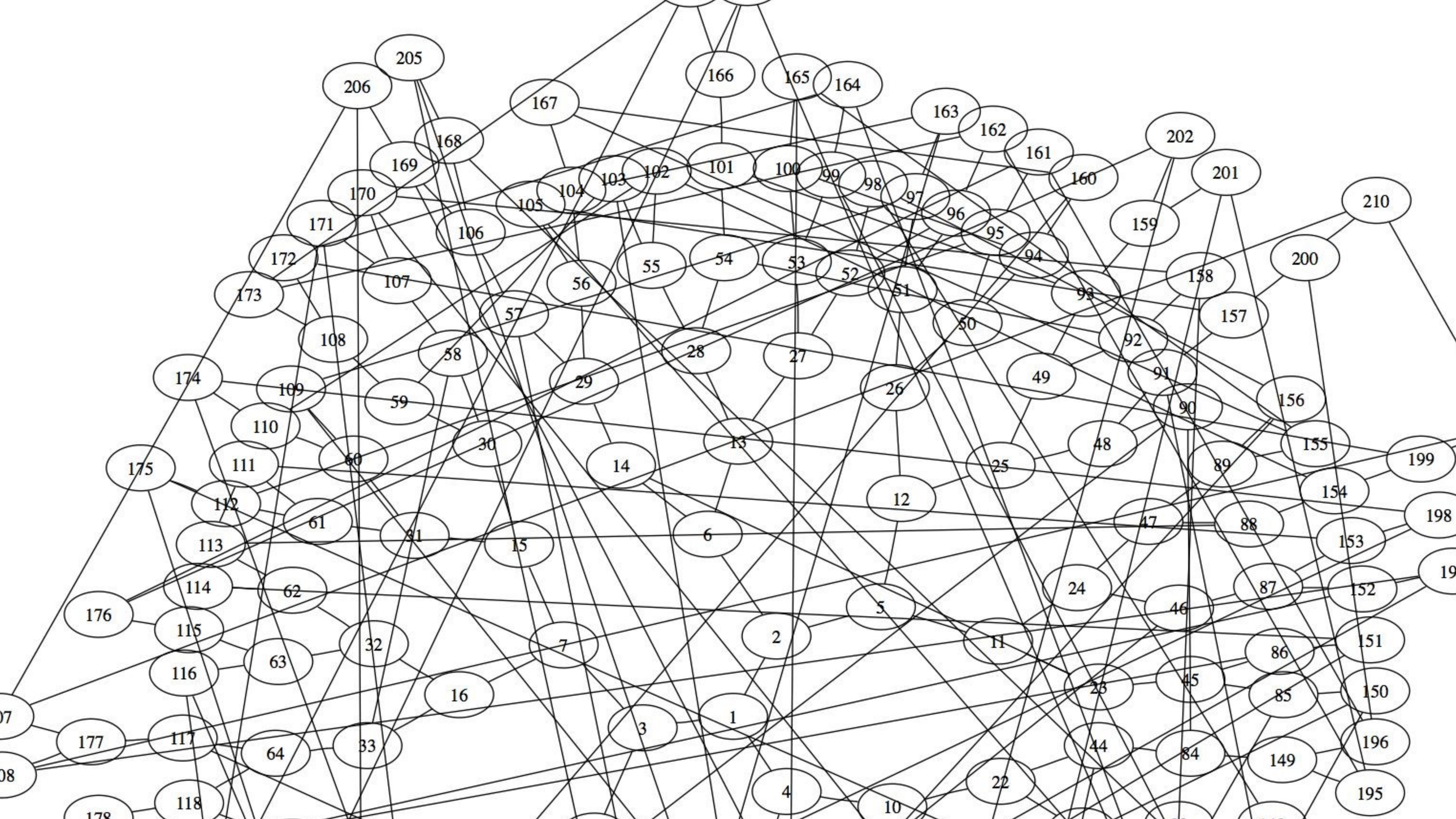
- Paths = isogenies of degree 2^n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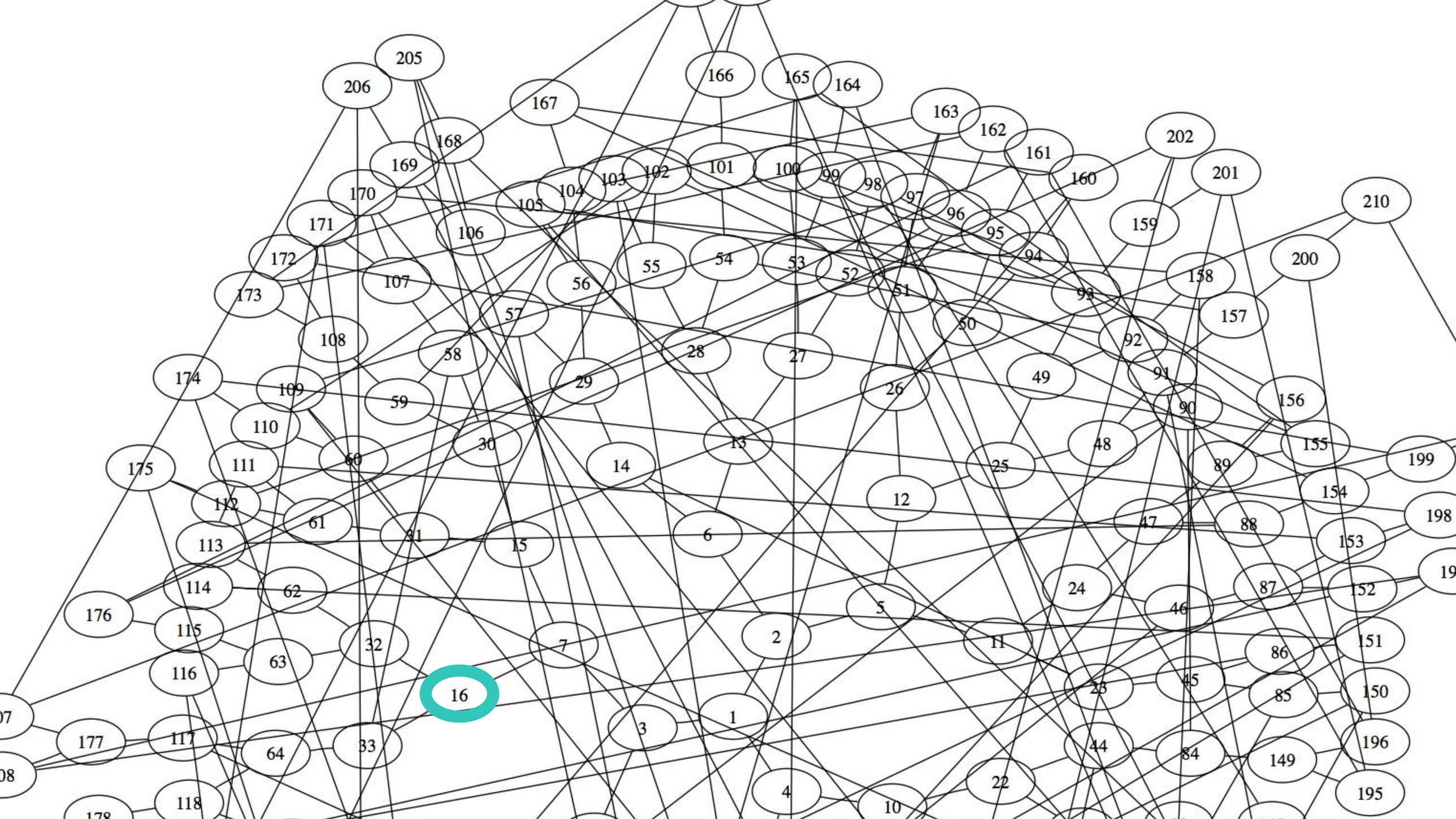
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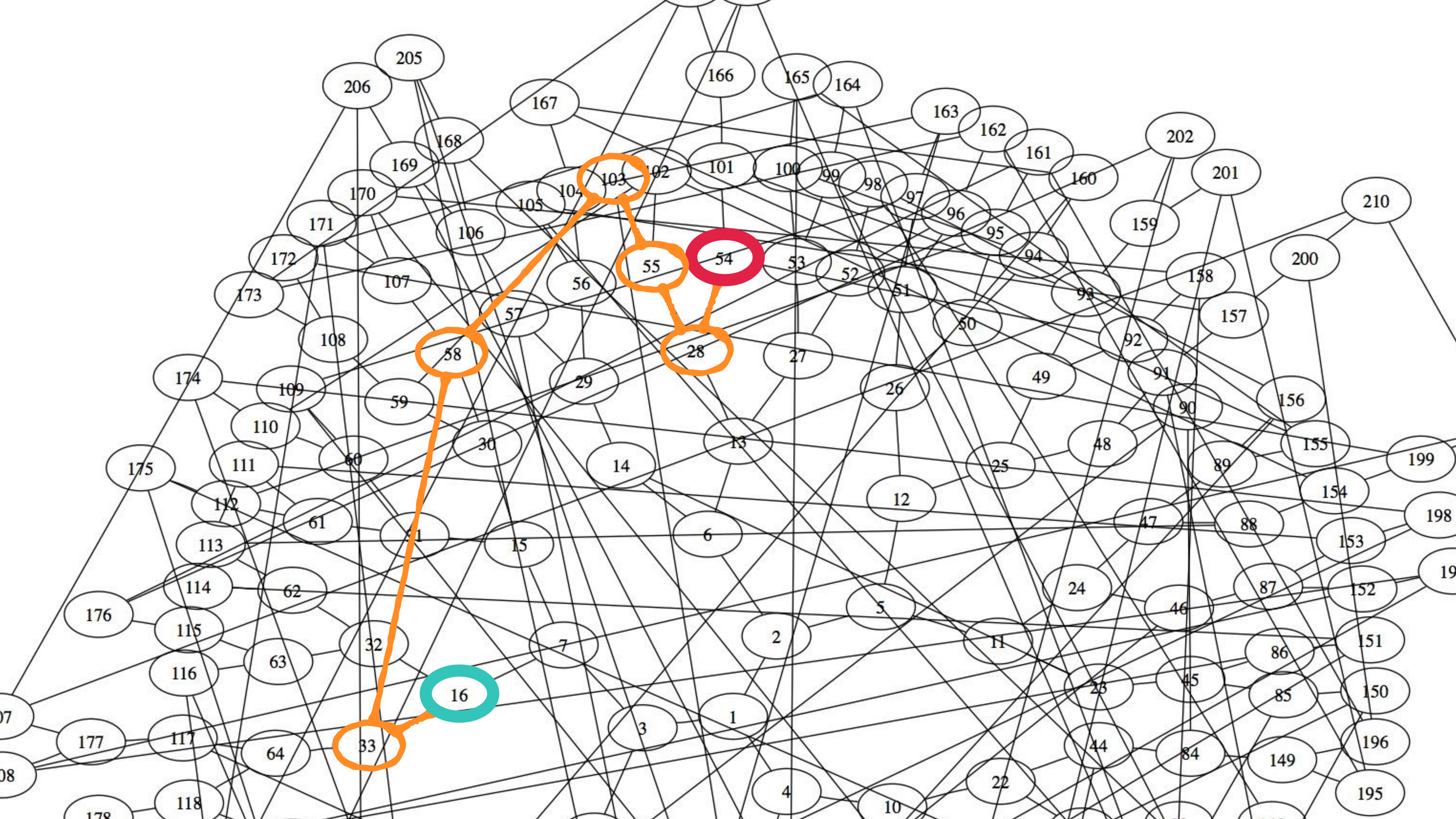
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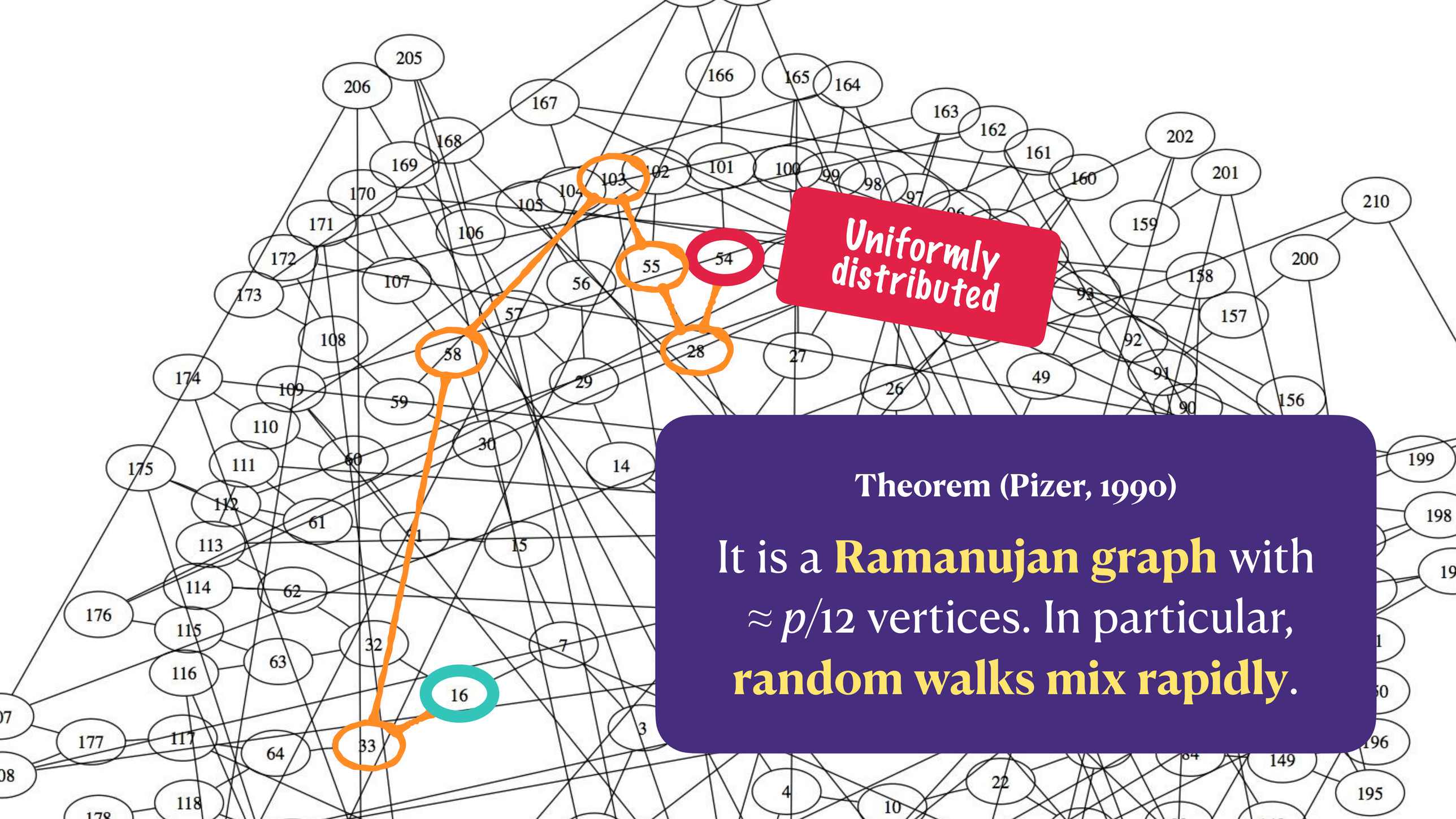
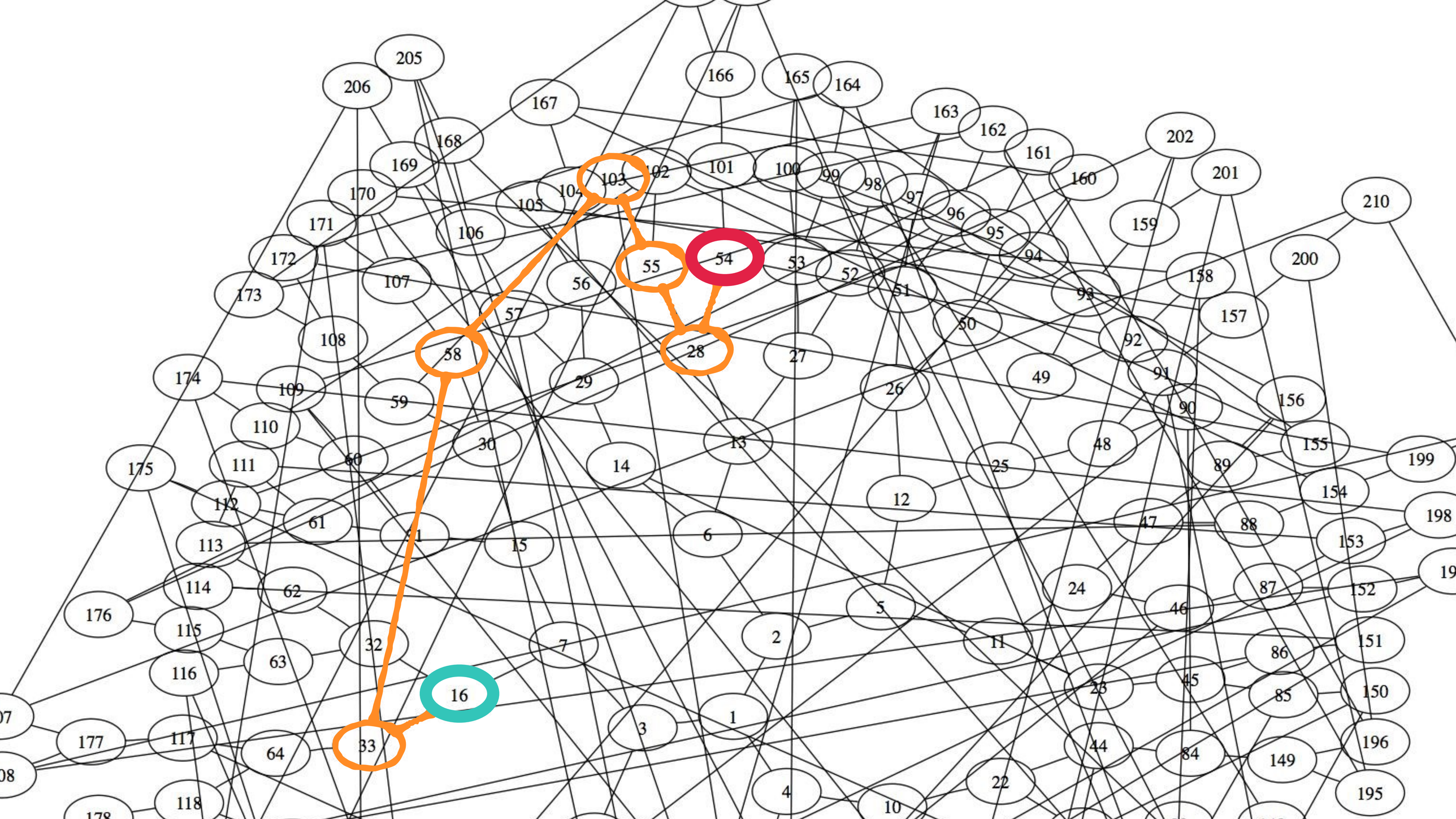


Theorem (Pizer, 1990)

It is a **Ramanujan graph** with $\approx p/12$ vertices. In particular, **random walks mix rapidly**.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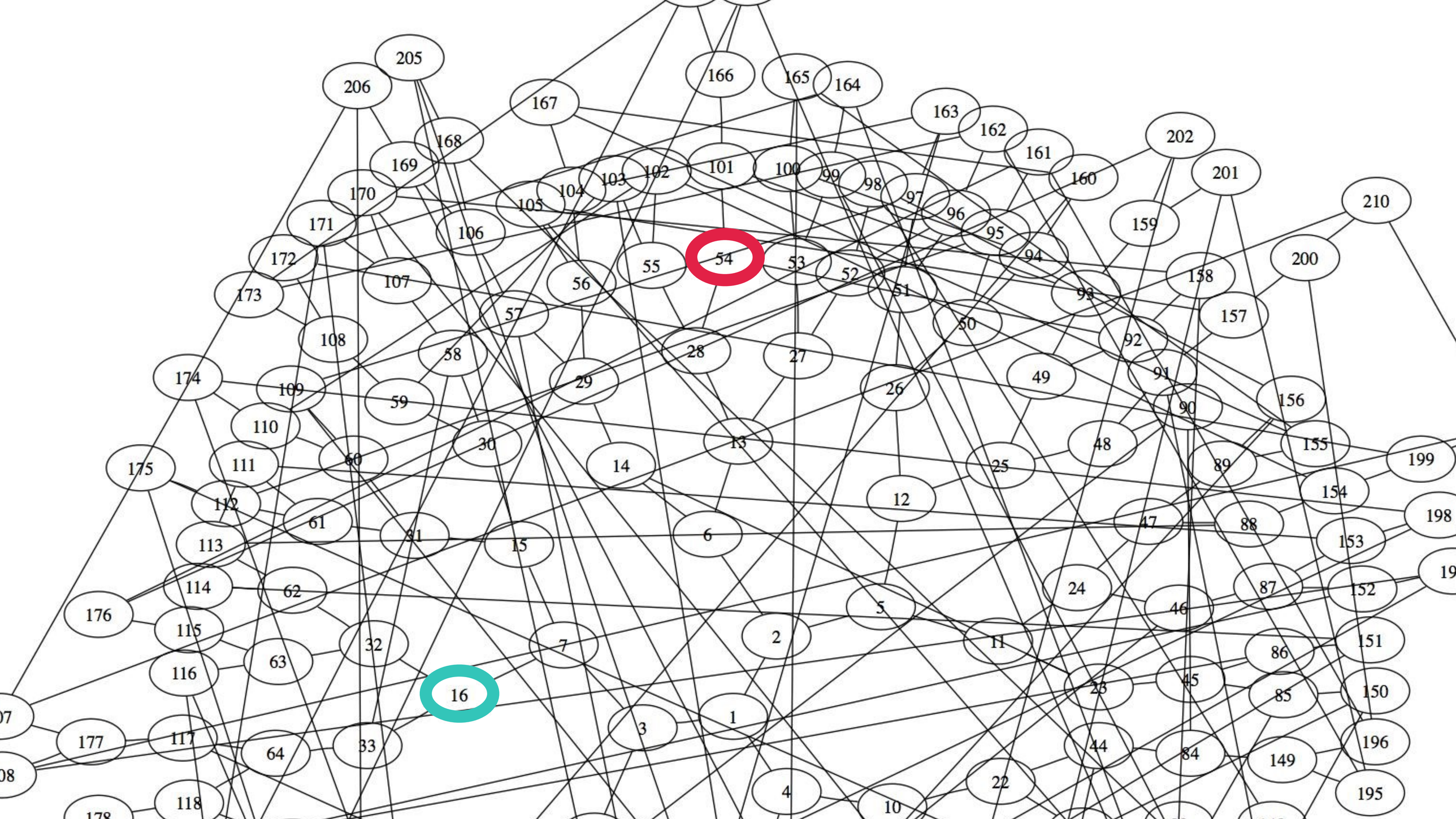
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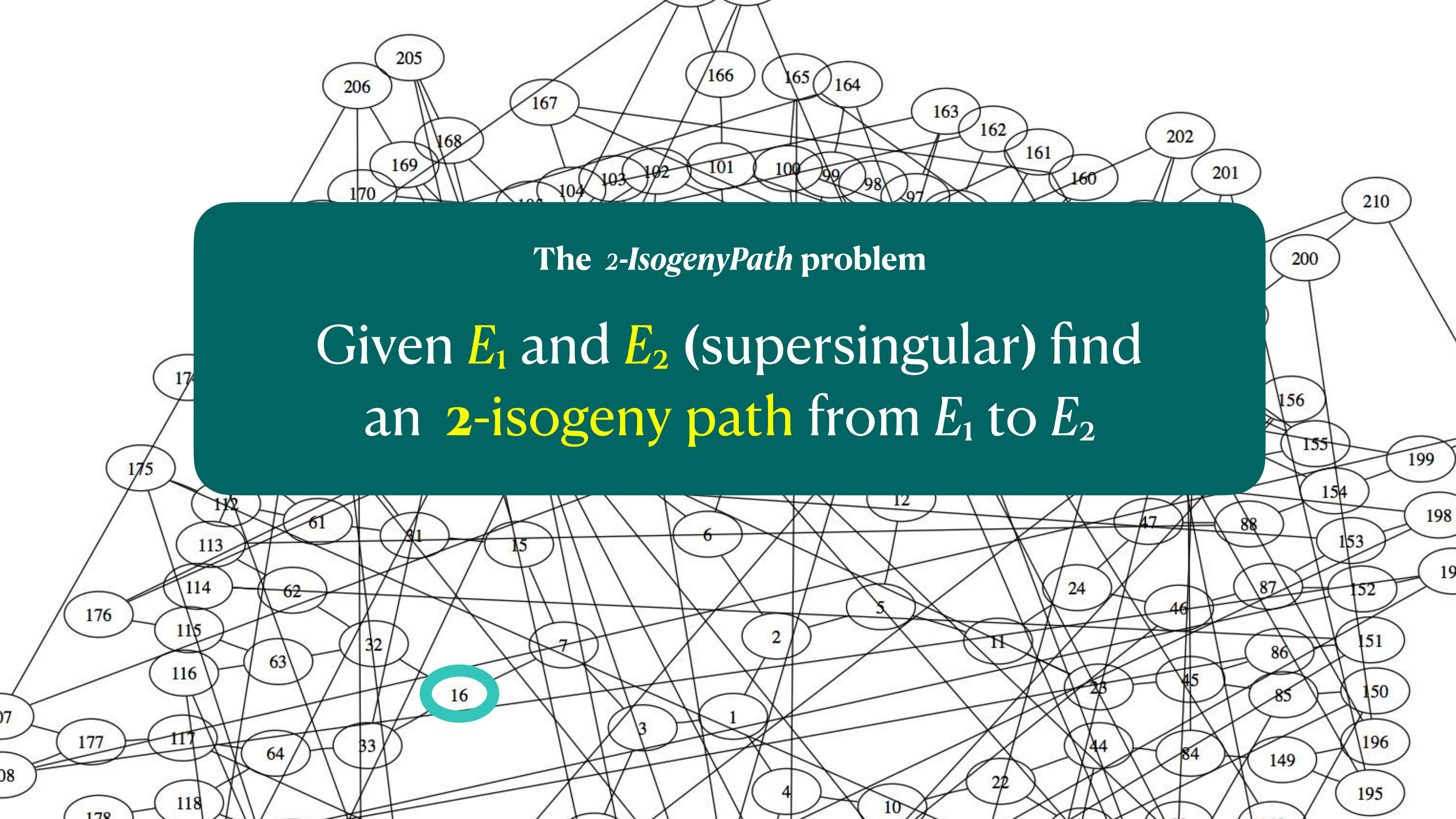
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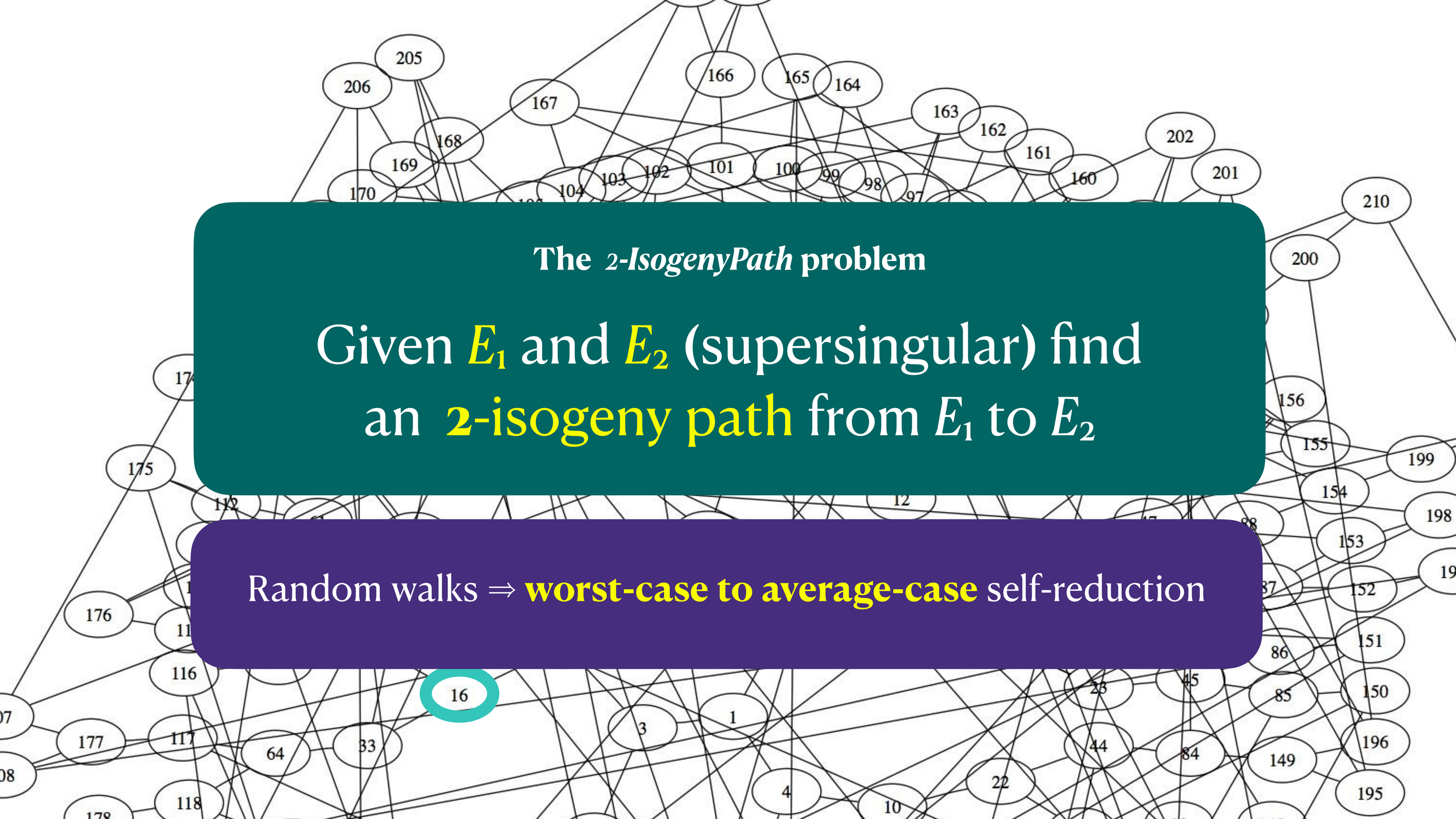

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Random walks ⇒ **worst-case to average-case** self-reduction

Interpolation

Representing arbitrary isogenies

Picture by Beppe Rijs

HD representation of isogenies

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a subgroup of E_1 of order 2^{2n}

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Somewhat fast, but requires $2^{2n} - d = a^2 + b^2$

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- Let (P, Q) is a basis of $E_1[2^n]$, with $2^{2n} > 4 \cdot \deg(\varphi)$
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a subgroup of E_1 of order 2^{2n}

Somewhat fast, but requires $2^{2n} - d = a^2 + b^2$

Fastest, but requires $2^{2n} - d = a^2$

Very costly, but always works

Interpolation: **Knowing φ on a few points $\Leftrightarrow$ knowing φ everywhere**

Corollary: $(d, P, Q, \varphi(P), \varphi(Q))$ is an efficient representation of φ, the "**interpolation representation**", or "**HD representation**"

Cost? evaluating an isogeny of degree 2^{2n} in dimension 2, 4 or 8

The *Isogeny* problem

Given E_1 and E_2 supersingular, find an isogeny $\varphi : E_1 \to E_2$ in interpolation representation

Isogeny

2-IsogenyPath


trivial
Isogeny
2-IsogenyPath

[Eisenträger, Hallgren, Lauter, Morrison, Petit – Eurocrypt 2018] / **[W. – FOCS 2021]** + **[Page, W. – Eurocrypt 2024]**

Endomorphisms

and computational problems

Picture by Beppe Rijs

Endomorphism ring

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Multiplication by $m \in \mathbb{Z}$ is an endomorphism

$$[m] : E \to E : P \mapsto P + \ldots + P$$

It forms a subring $\mathbb{Z} \subset \mathrm{End}(E)$

Endomorphism ring

What is the structure of End(E)?

- It contains $\mathbb{Z} \subset \mathrm{End}(E)$...

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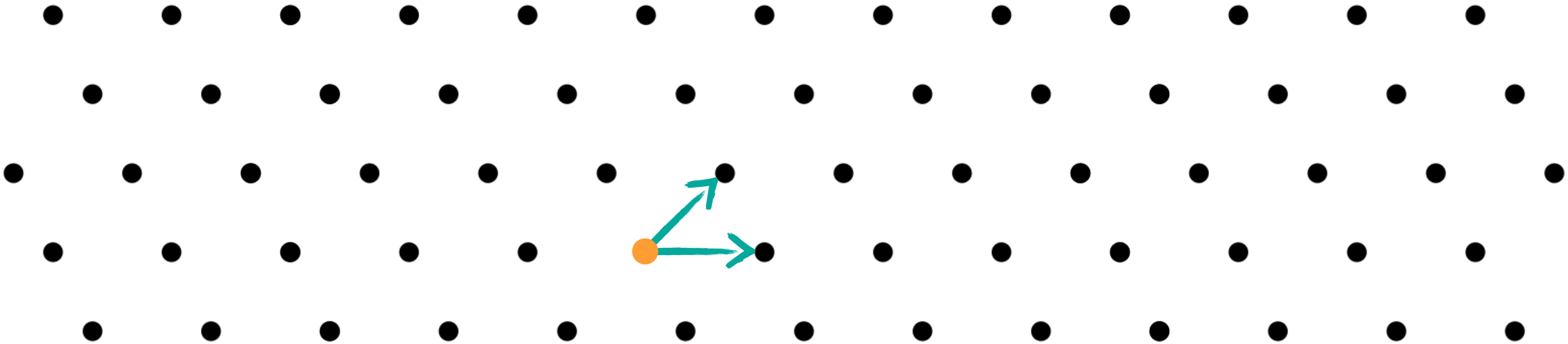

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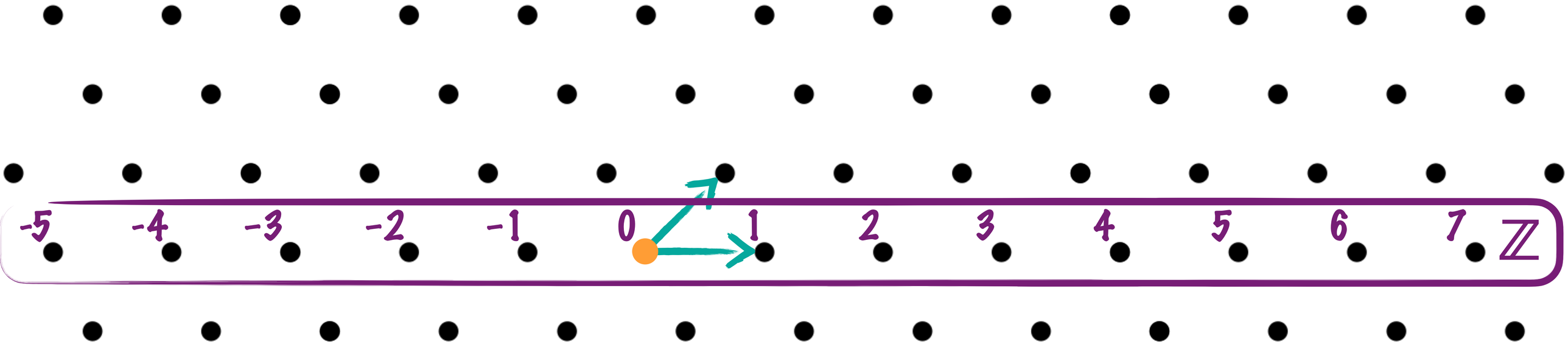

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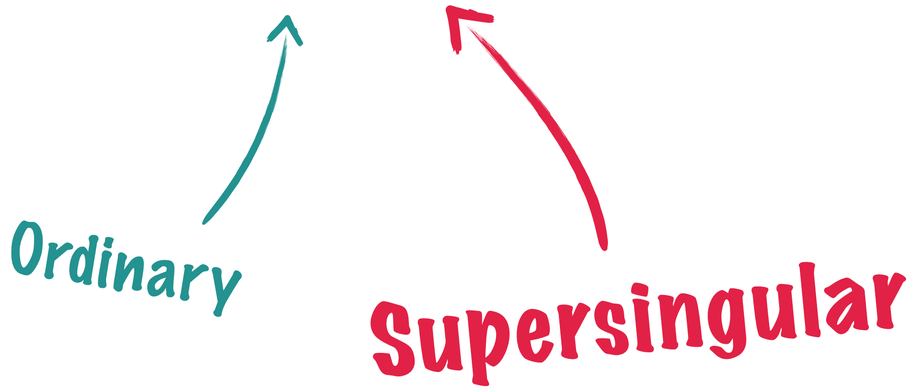

The endomorphism ring problem

For E **supersingular** $\mathrm{End}(E) = \{\varphi : E \to E\}$ is a lattice of dimension 4

The *EndRing* problem

Given E (supersingular) find 4 generators of the endomorphism ring $\mathrm{End}(E)$

Example

Example: $p \equiv 3 \pmod 4$, so $\mathbb{F}_{p^2} = \mathbb{F}_p(\alpha)$ where $\alpha^2 = -1$, and

Consider $E_0 : y^2 = x^3 + x$

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EndRing

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The **quaternion algebra $B_{p,\infty}$** is the ring (for $p \equiv 3 \pmod 4$)

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$$\simeq \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\,\frac{i + j}{2} \oplus \mathbb{Z}\,\frac{1 + ij}{2} \subset B_{p,\infty}$$

EndRing

MaxOrder

One Endomorphism

- We always have $\mathbb{Z} \subset \mathrm{End}(E)$, that part is easy
- Finding any **non-scalar** endomorphism?

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The *OneEnd* problem

Given E (supersingular) find ***one*** endomorphism $\alpha \in \mathrm{End}(E) \setminus \mathbb{Z}$

Which is hardest? Easiest?

EndRing

2-IsogenyPath

OneEnd

Isogeny

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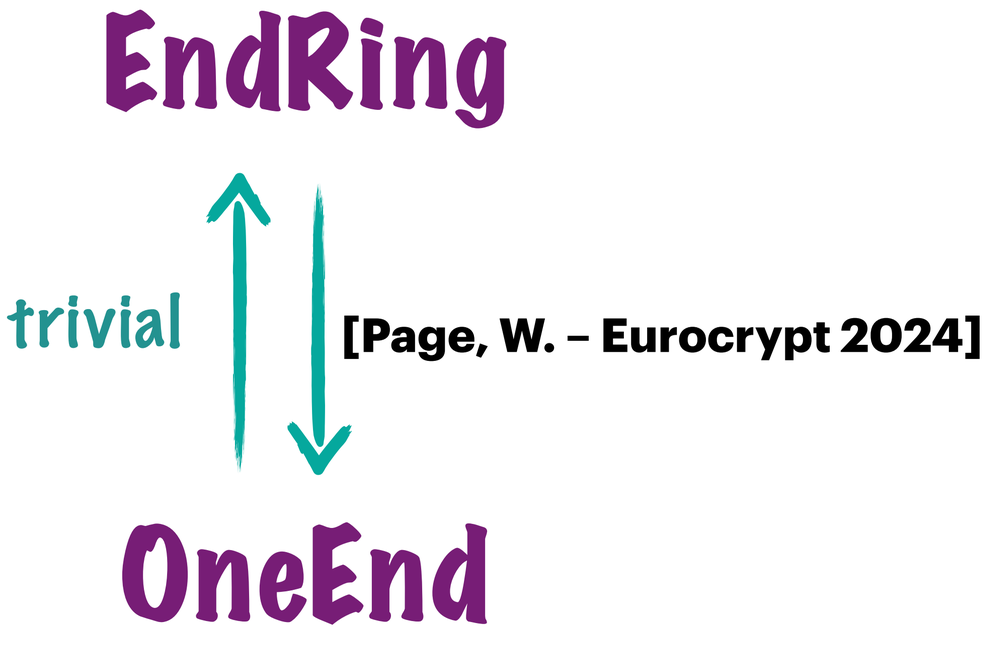

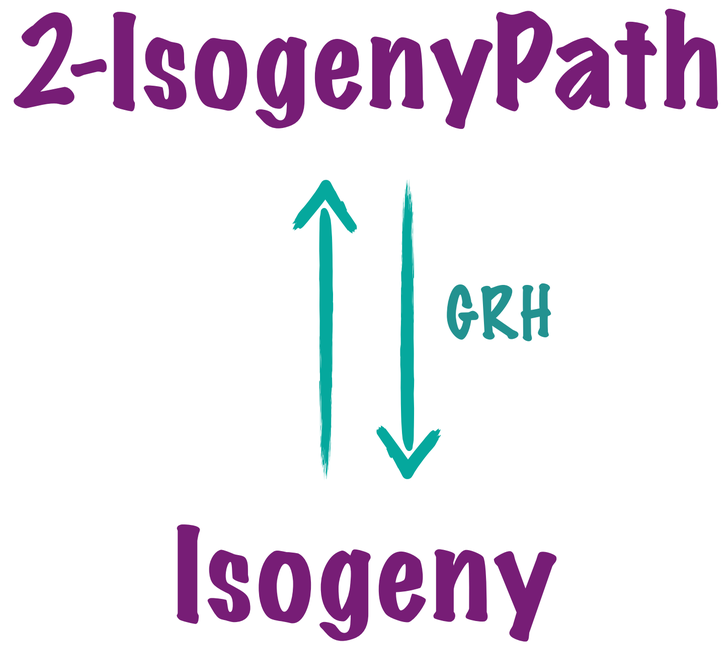

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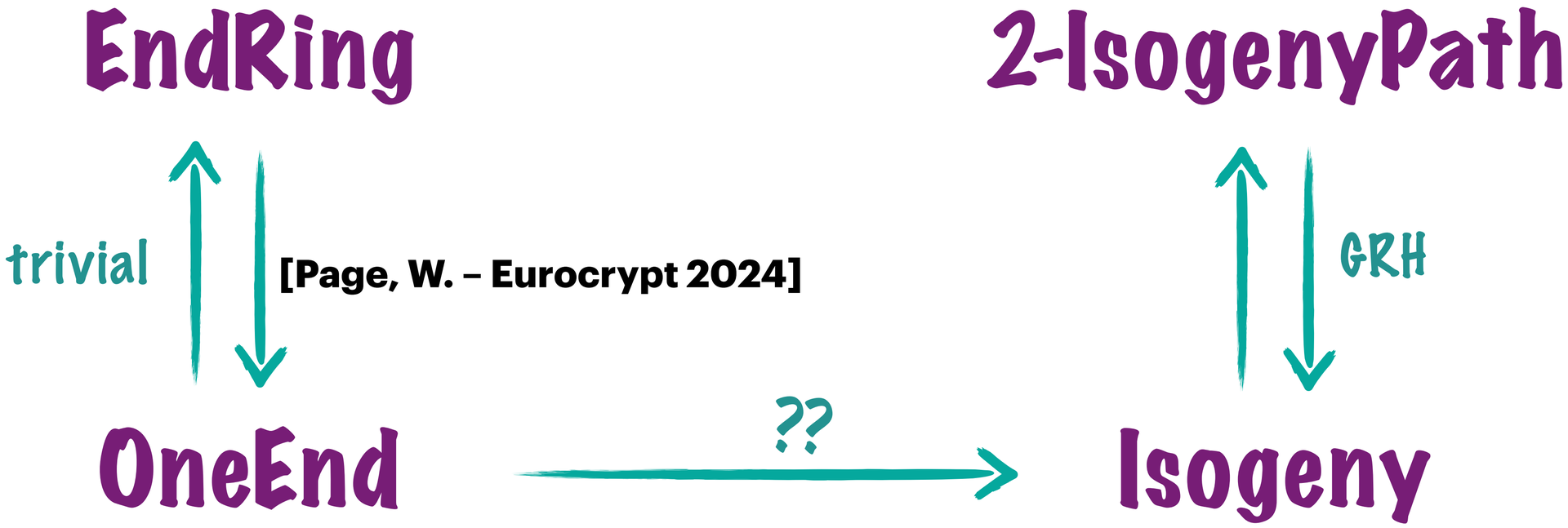

OneEnd reduces to Isogeny

Suppose we can solve **Isogeny**.
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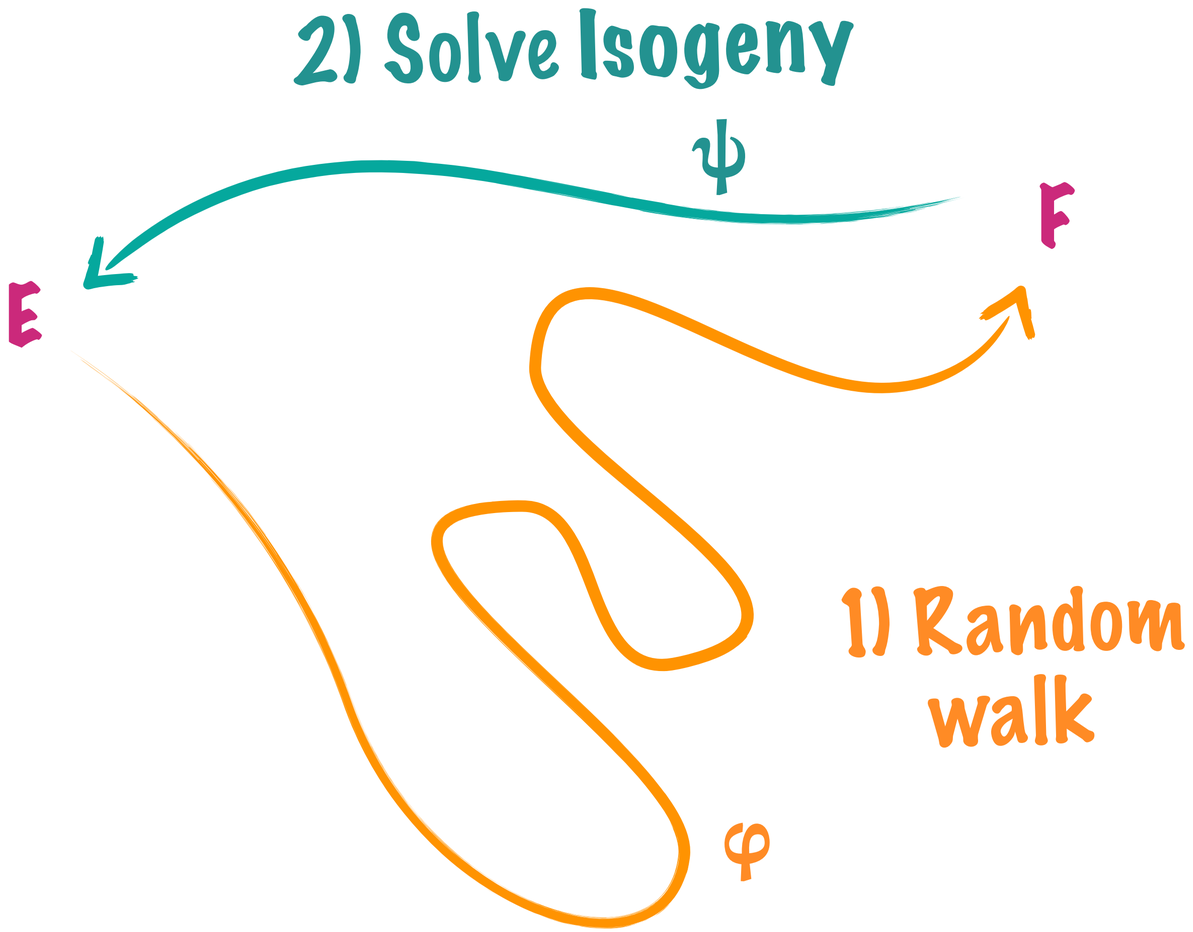

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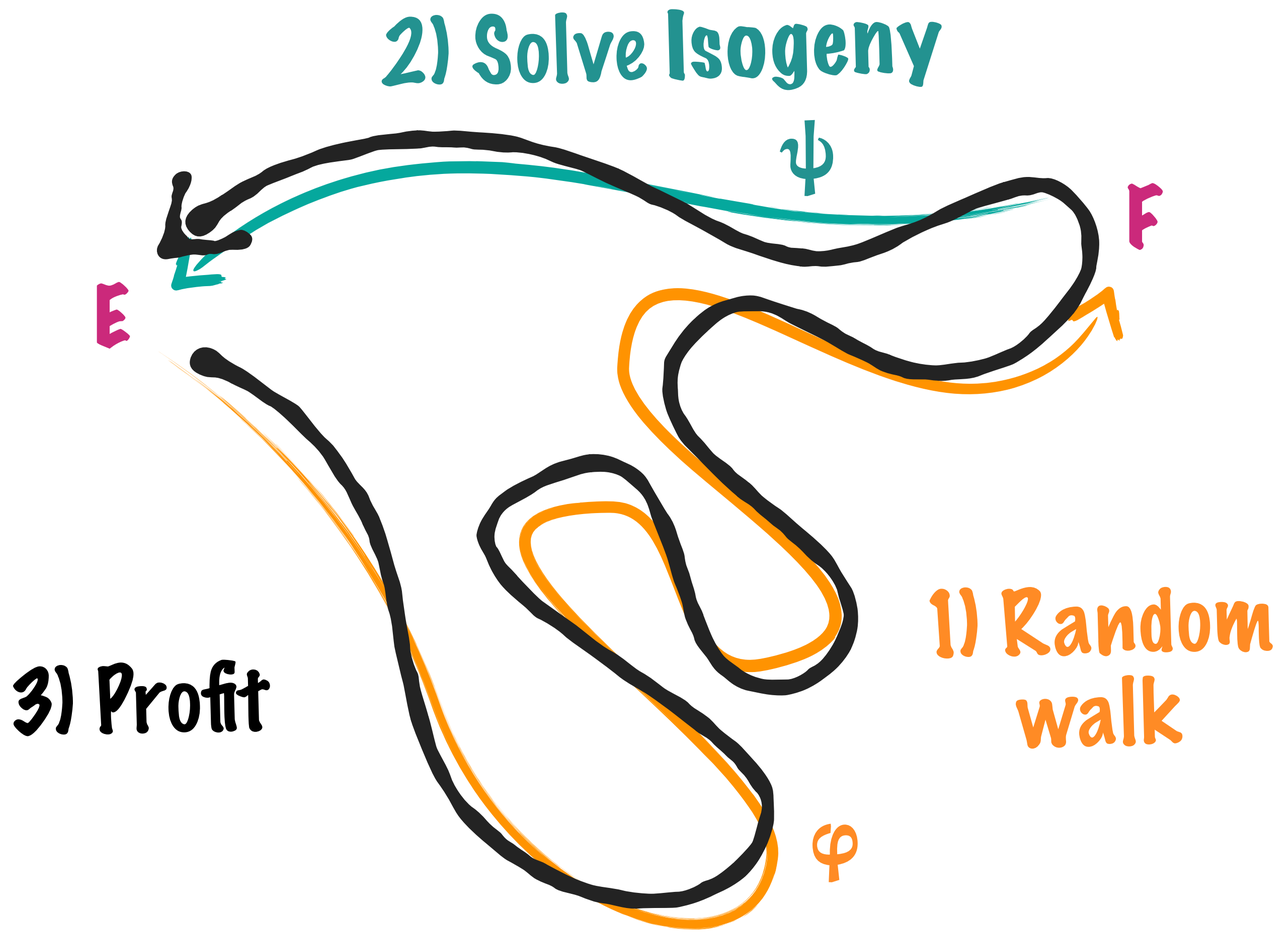

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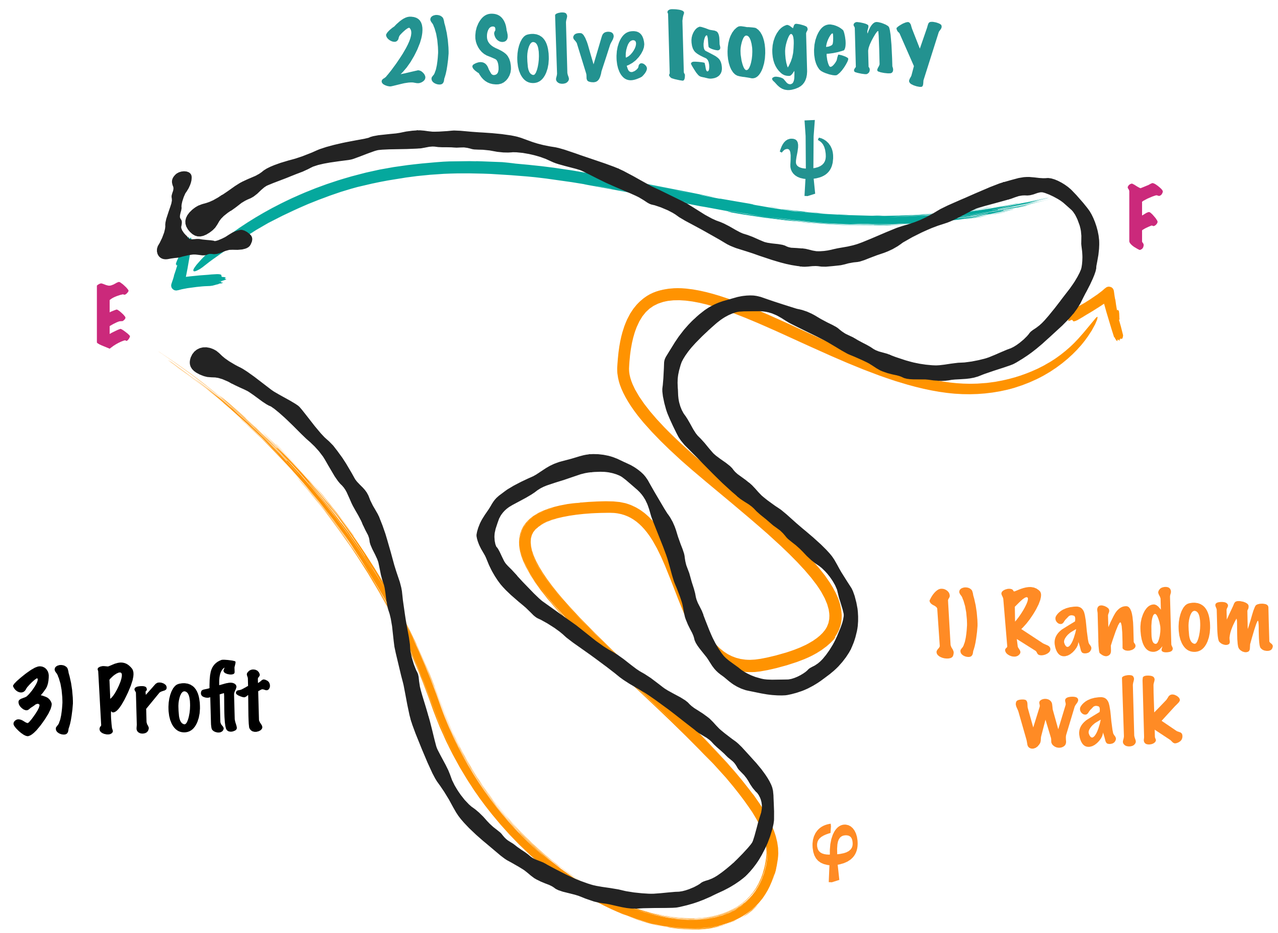
How to find endomorphisms of E:

- If φ is long enough $\psi \circ \varphi \notin \mathbb{Z}$



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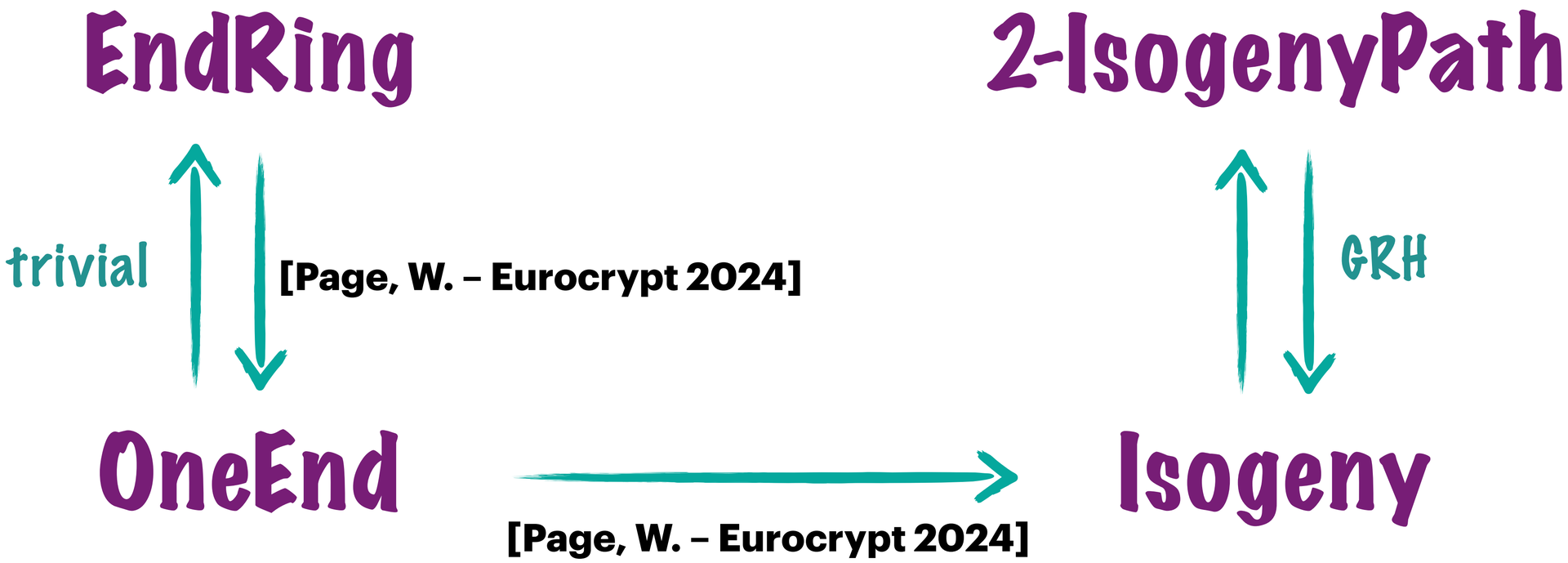

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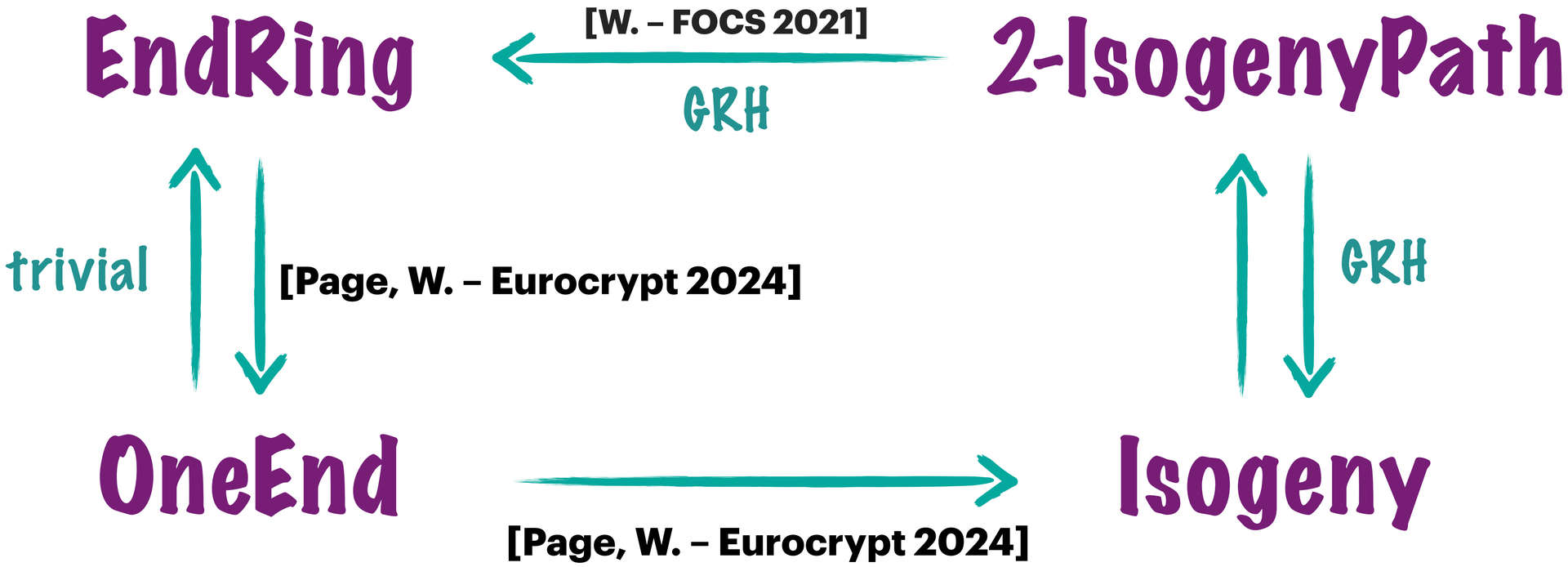

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Quaternion World

Deuring correspondence

Supersingular curves E over $\mathbb{F}_{p2}$
(up to isomorphism)

Maximal orders $\mathcal{O}$ in $B_{p,\infty}$

$\mathcal{O} \simeq \mathbf{End}(E)$

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KLPT, Clapoti...

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End(–) is a GPS that allows you to find your way between supersingular curves:

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Computing End(–) ⇔ Computing Hom(–, –)

Key generation

Generating a curve with its endomorphism ring

Picture by Beppe Rijs

Keys

- **Public key**: a supersingular curve E_{pk}
- **Secret key**: a basis of $\text{End}(E_{\text{pk}})$

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How to generate a random E_{pk} together with $\text{End}(E_{\text{pk}})$?

How to generate even a single supersingular curve?

A special supersingular curve

Example: $p \equiv 3 \pmod 4$, so $\mathbb{F}_{p^2} = \mathbb{F}_p(\alpha)$ where $\alpha^2 = -1$, and

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- $\iota : E_0 \to E_0 : (x, y) \mapsto (-x, \alpha y)$ $\quad \iota^2 = [-1]$
- $\pi : E_0 \to E_0 : (x, y) \mapsto (x^p, y^p)$ $\quad \iota\pi = -\pi\iota$

$$\textbf{End}(E_0) = \mathbb{Z} \oplus \mathbb{Z}\iota \oplus \mathbb{Z}\,\frac{\iota + \pi}{2} \oplus \mathbb{Z}\,\frac{1 + \iota\pi}{2}$$

A special supersingular curve

Example: $p \equiv 3 \pmod 4$, so $\mathbb{F}_{p^2} = \mathbb{F}_p(\alpha)$ where $\alpha^2 = -1$, and

Consider $E_0 : y^2 = x^3 + x$

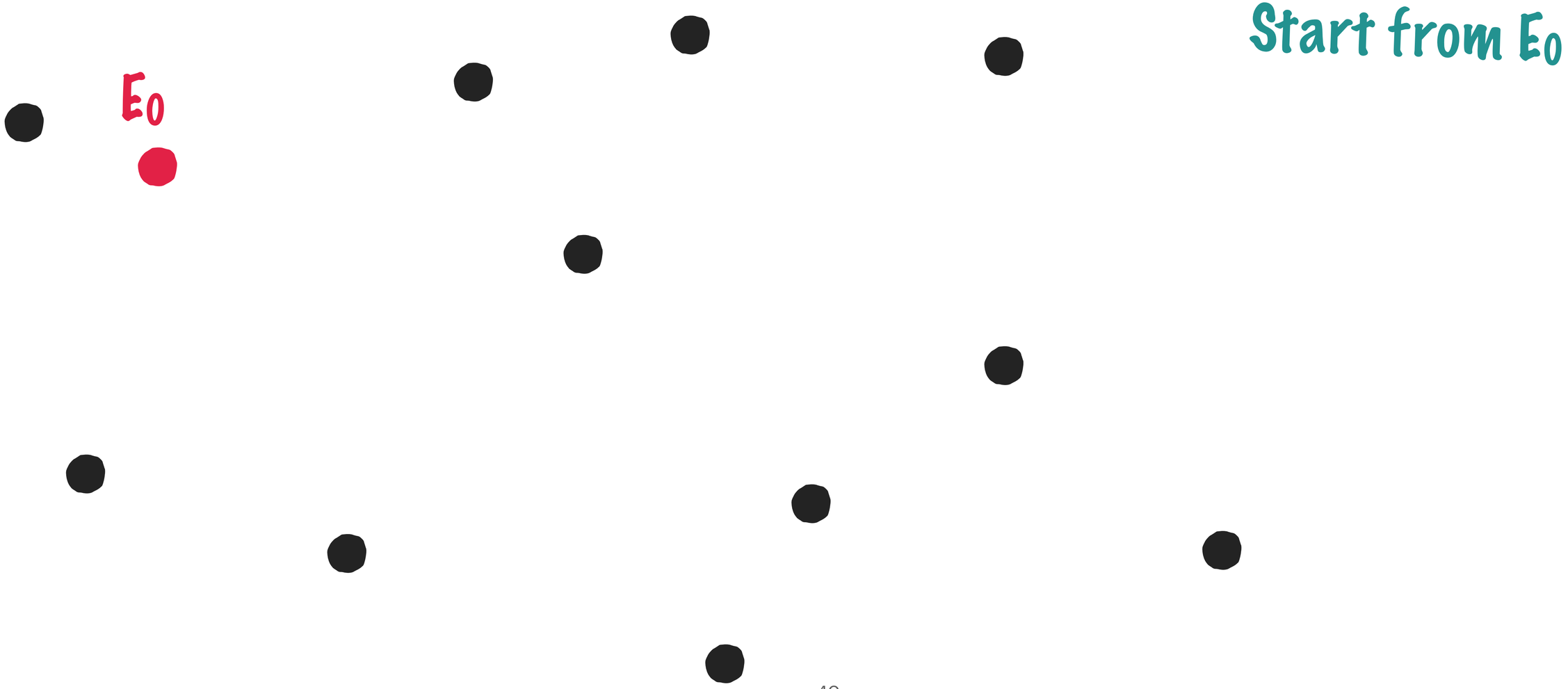
Two non-trivial endomorphisms:

- $\iota : E_0 \to E_0 : (x, y) \mapsto (-x, \alpha y)$ $\quad \iota^2 = [-1]$
- $\pi : E_0 \to E_0 : (x, y) \mapsto (x^p, y^p)$ $\quad \iota\pi = -\pi\iota$

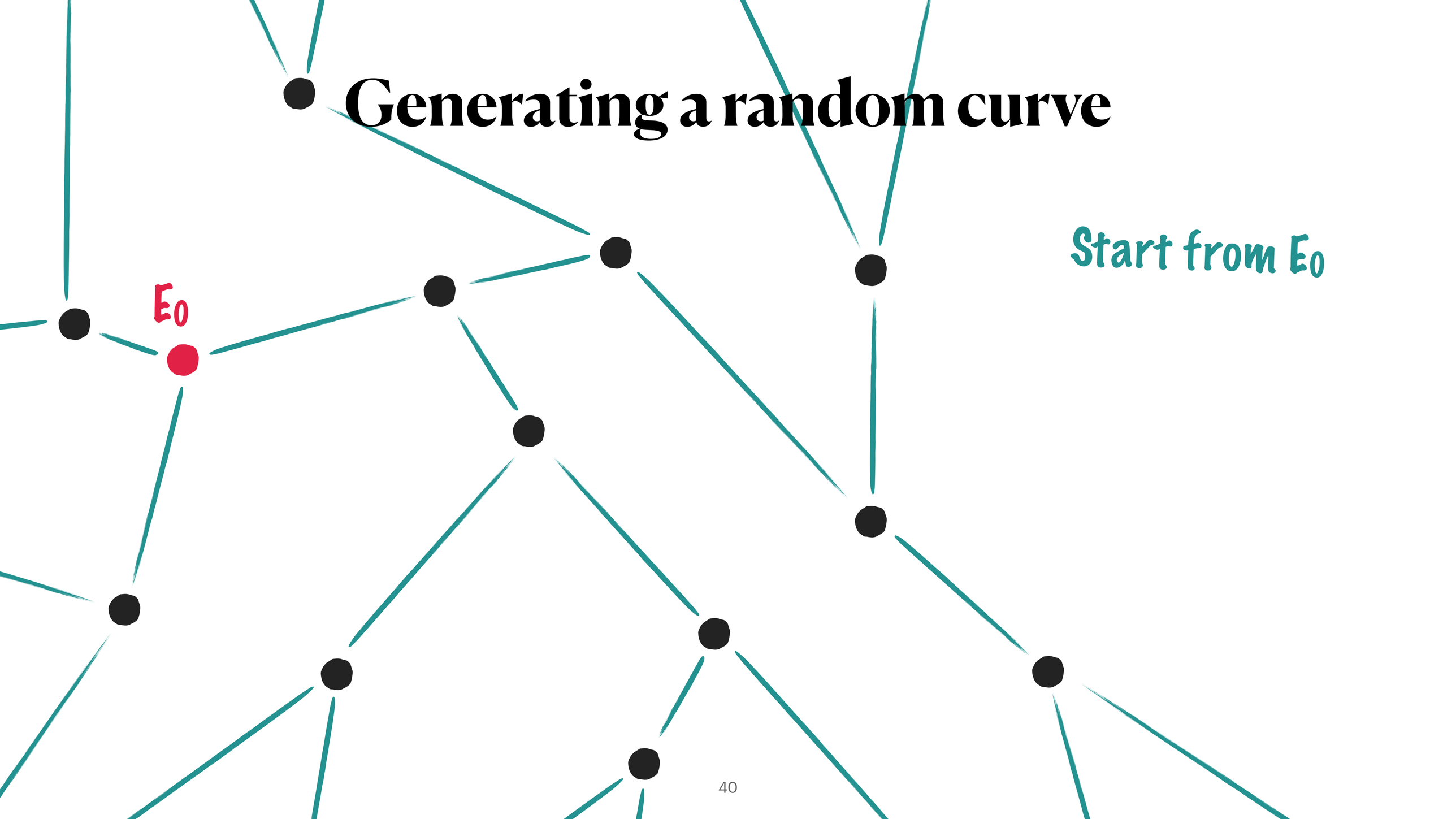
$$\textbf{End}(E_0) = \mathbb{Z} \oplus \mathbb{Z}\iota \oplus \mathbb{Z}\,\frac{\iota + \pi}{2} \oplus \mathbb{Z}\,\frac{1 + \iota\pi}{2}$$

E_0 and End(E_0) is our reference

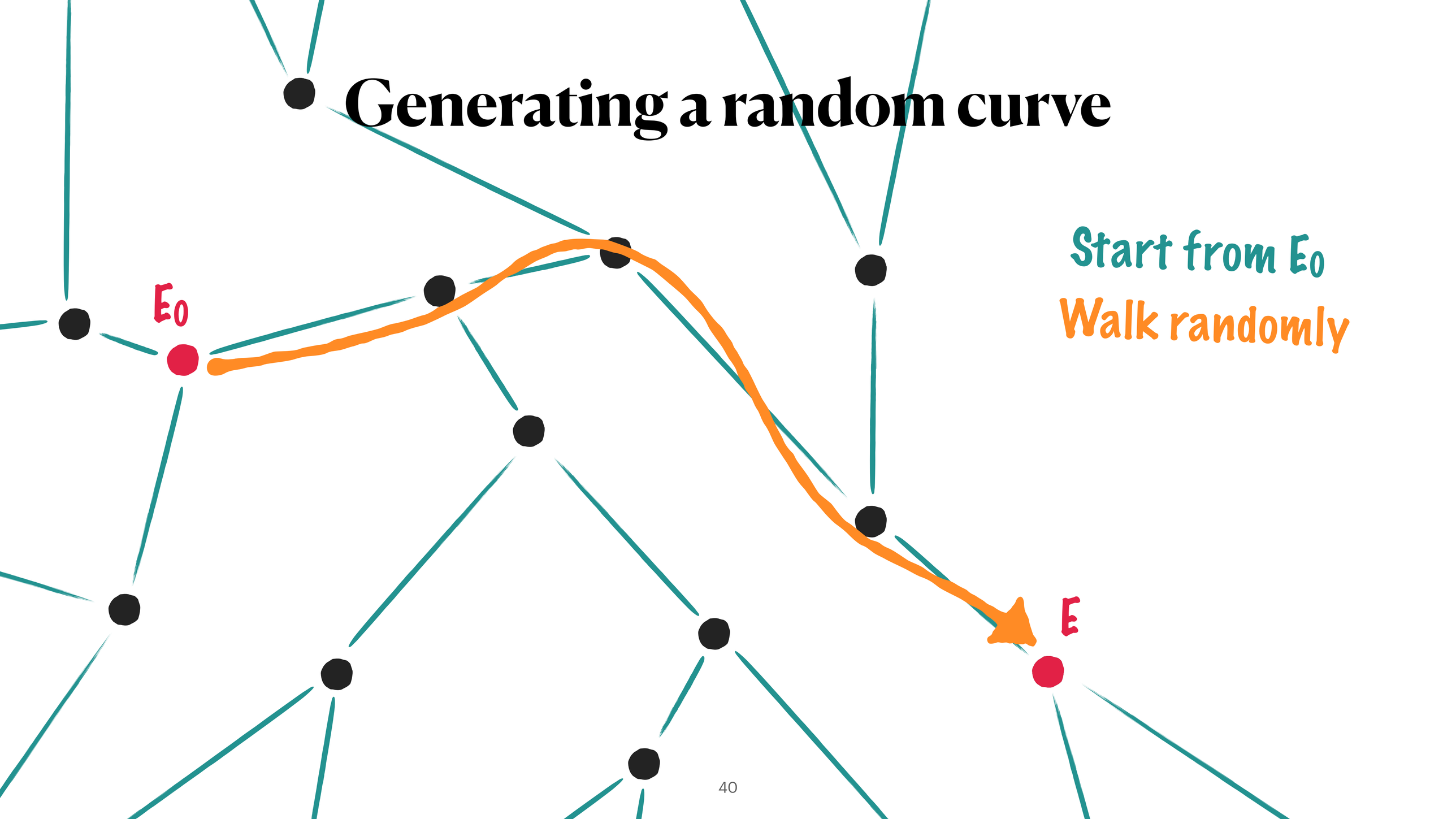
Generating a random curve



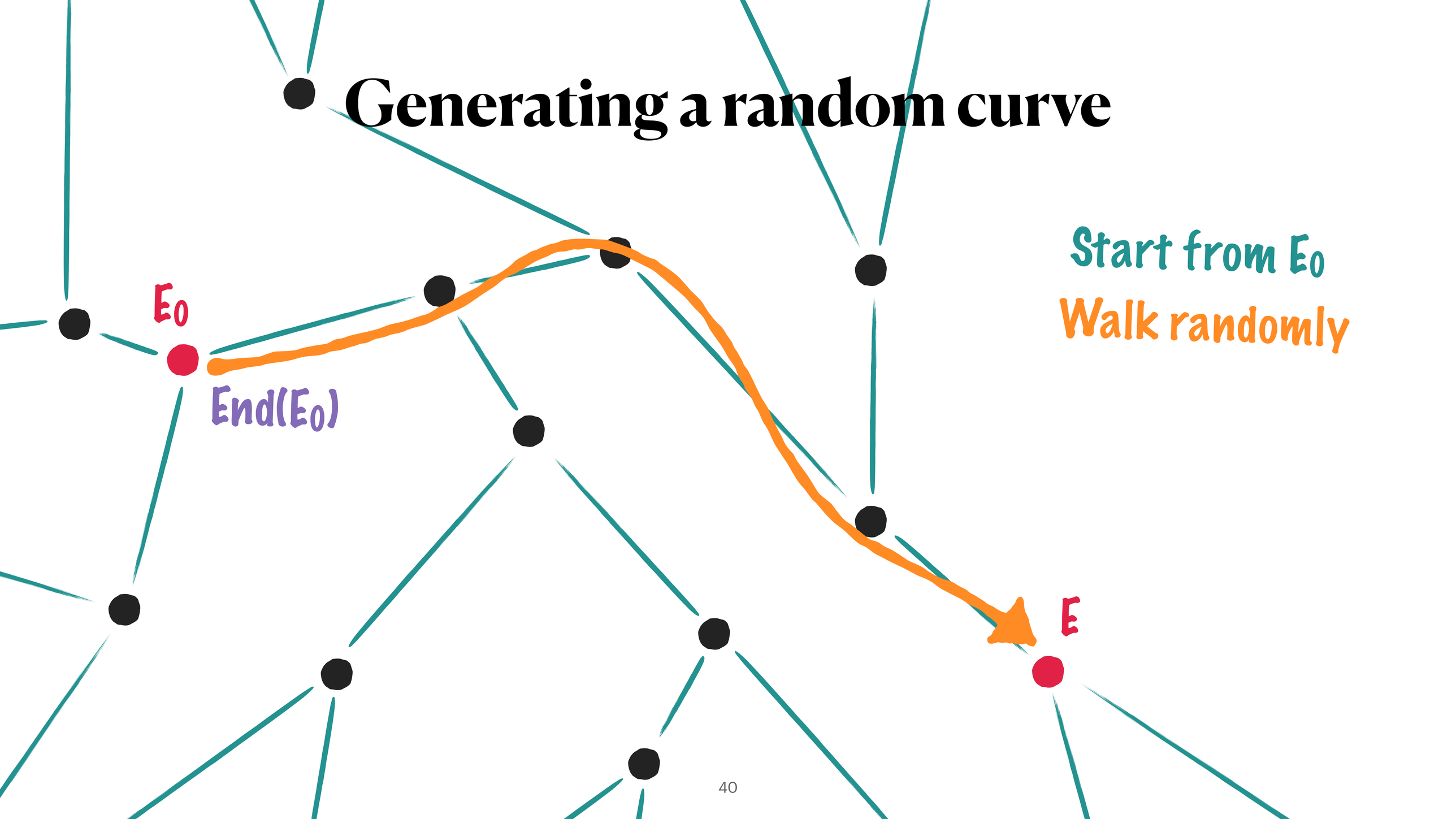
Generating a random curve



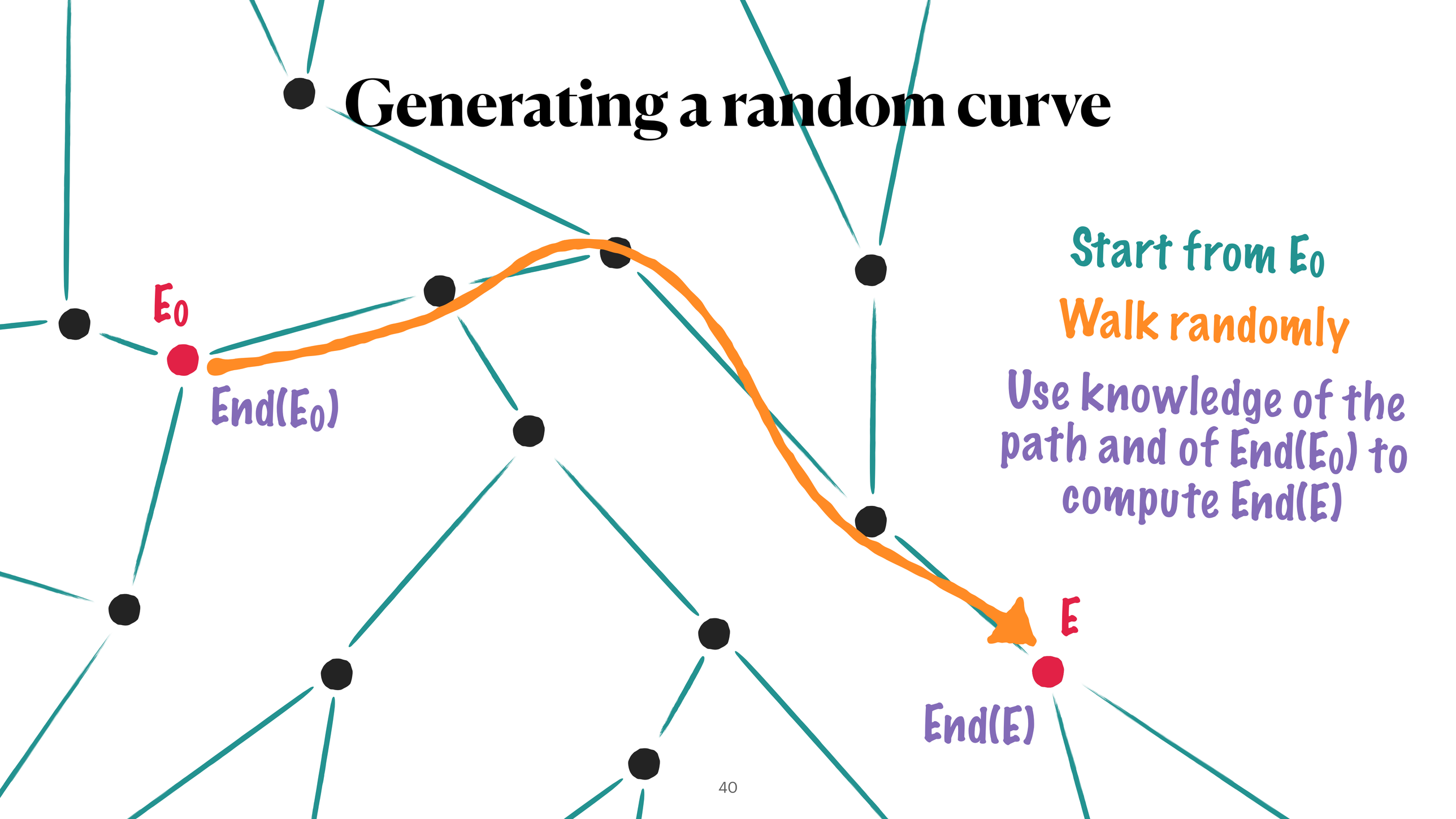
Generating a random curve



Generating a random curve



Generating a random curve



One can generate $(E, \mathrm{End}(E))$ with E uniform