Computing two-dimensional isogenies for SQIsign

Maria Corte-Real Santos
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I. Motivation

A new tool

Interpolation

Given coprime integers $N < N'$ and four points P_1, P_2, Q_1, Q_2 such that $\langle P_1, P_2 \rangle = E_1[N']$ and $\langle Q_1, Q_2 \rangle = E_2[N']$ with N' a B-powersmooth integer.

One can **efficiently check existence** and **compute** the isogeny of degree N

$$\varphi : E_1 \to E_2 \text{ such that } \varphi(P_1) = Q_1 \text{ and } \varphi(P_2) = Q_2.$$

Damien Robert. *On the efficient representation of isogenies* (2024).

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$\varphi : E_1 \to E_2$ such that $\varphi(P_1) = Q_1$ and $\varphi(P_2) = Q_2$.

Requires computing dimension ≥ 2 isogenies of degree ℓ between products of elliptic curves.

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SQIsign

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SQIsign2D–West

The Fast, the Small, and the Safer

Andrea Basso[1,2][0000–0002–3270–1069], Pierrick Dartois[3,4][0009–0008–2808–9867], Luca De Feo[2][0000–0002–9321–0773], Antonin Leroux[5,6][0009–0002–3737–0075], Luciano Maino[1][0009–0005–4495–5102], Giacomo Pope[1,7], Damien Robert[3,4][0000–0003–4378–4274], and Benjamin Wesolowski[8][0000–0003–1249–6077]

SQIsign2D-East: A New Signature Scheme Using 2-dimensional Isogenies

Kohei Nakagawa[1] and Hiroshi Onuki[2]

SQIPrime: A dimension 2 variant of SQISignHD with non-smooth challenge isogenies

Max Duparc and Tako Boris Fouotsa

SQIsign

- New variants of SQIsign exploit this new tool.

- For a sufficiently nice set-up, we can restrict to computing 2D isogenies.

- Two-dimensional isogenies are used in key generation, signing *and* verification!

Our goal

$$\Phi : E_1 \times E_2 \longrightarrow E_3 \times E_4$$

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Inputs

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Primes $p \neq \ell$

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Primes $p \neq \ell$

Domain $E_1 \times E_2$ defined over $\bar{\mathbb{F}}_p$

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Inputs

Primes $p \neq \ell$

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Degree ℓ^k

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$$\Phi : E_1 \times E_2 \longrightarrow E_3 \times E_4$$

Inputs

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Degree ℓ^k

$R, S \in E_1 \times E_2[\ell^k]$ generating kernel

Our goal

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$\xrightarrow{\Phi}$

Outputs

Codomain $E_3 \times E_4$

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Points $P_1, \ldots, P_n \in E_1 \times E_2$

$$\xrightarrow{\Phi}$$

Outputs

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Outputs

Codomain $E_3 \times E_4$

Image points $\Phi(P_1), \ldots, \Phi(P_n)$

Our goal

$$\Phi : E_1 \times E_2 \longrightarrow E_3 \times E_4$$

Inputs

Cryptographic sized p

Domain $E_1 \times E_2$ defined over $\bar{\mathbb{F}}_p$

Degree 2^k

$R, S \in E_1 \times E_2[2^k]$ generating kernel

Points $P_1, \ldots, P_n \in E_1 \times E_2$

$\xrightarrow{\Phi}$

Outputs

Codomain $E_3 \times E_4$

Image points $\Phi(P_1), \ldots, \Phi(P_n)$

Structure of Talk

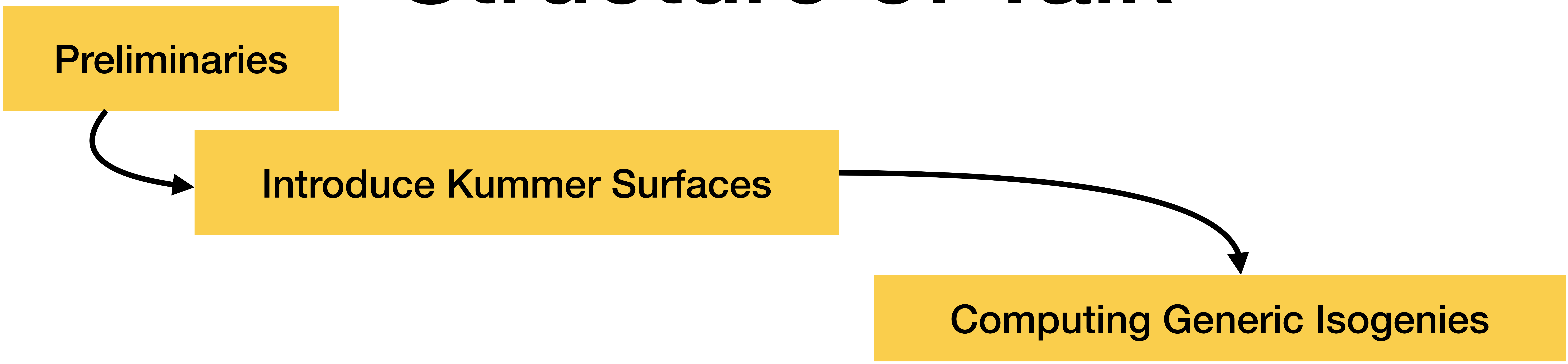
Preliminaries

Structure of Talk

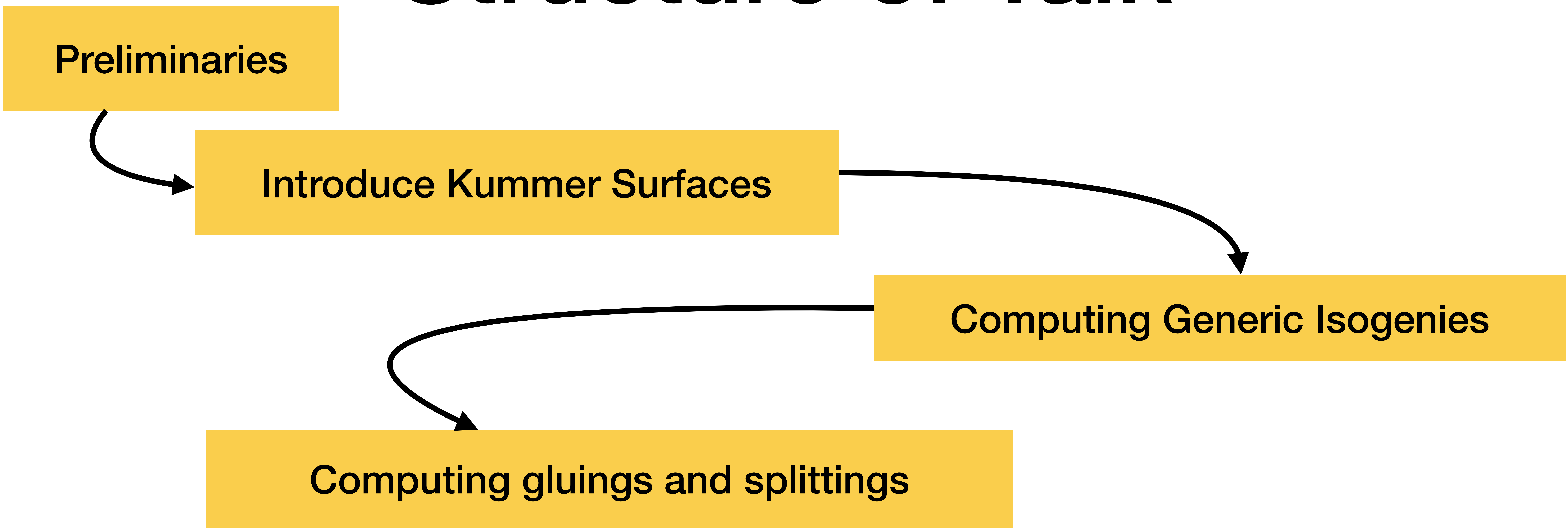
Preliminaries

Introduce Kummer Surfaces

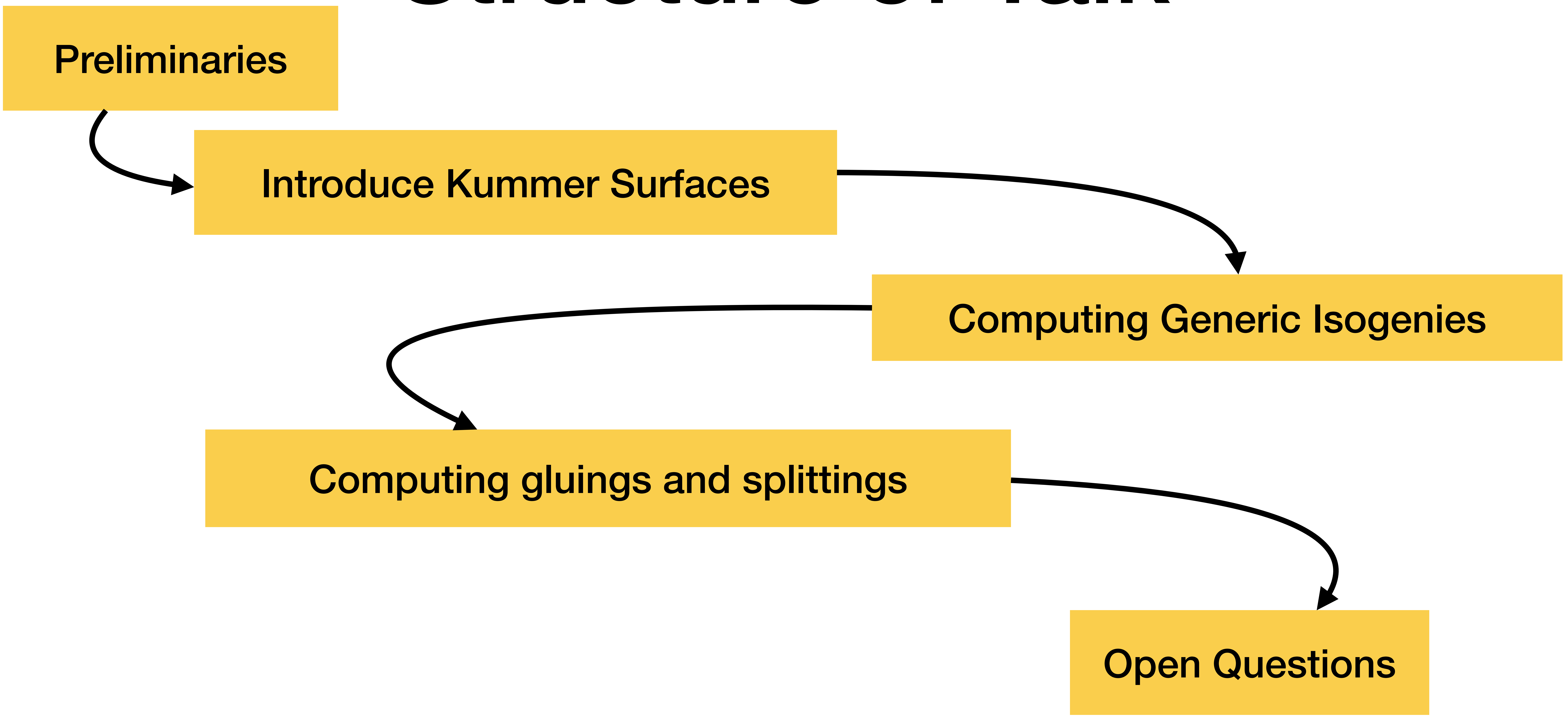
Structure of Talk



Structure of Talk



Structure of Talk



II. Preliminaries

Moving to higher dimension

We now consider **superspecial principally polarised abelian surfaces** (A, λ).

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gives you an embedding
into projective space

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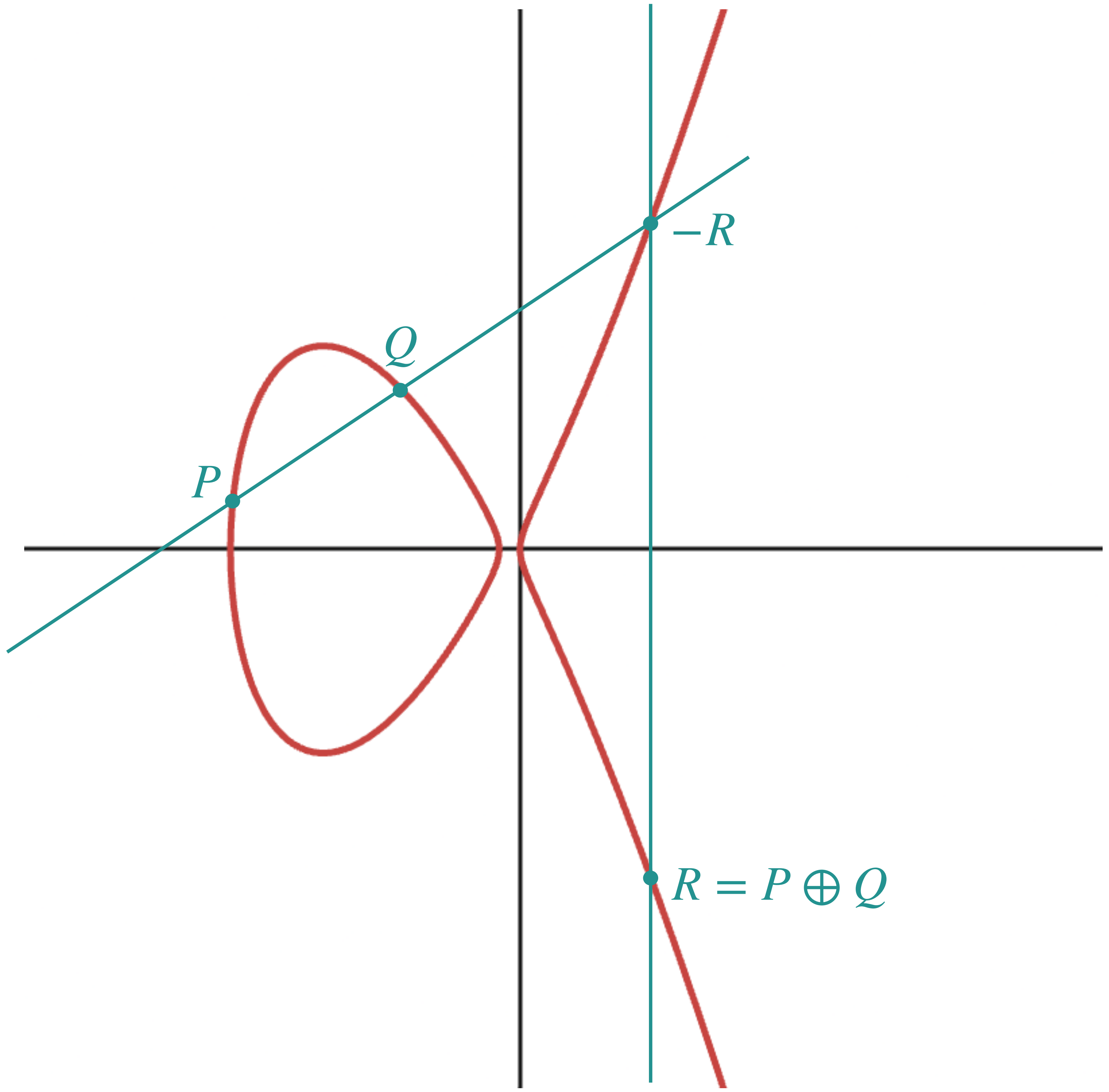
There are two types:

- Products of elliptic curves $E_1 \times E_2$
- Jacobians J_C of genus 2 hyperelliptic curves C

Moving to higher dimension

Elliptic Curves

$$E\colon y^2 = x^3 + Ax + B$$



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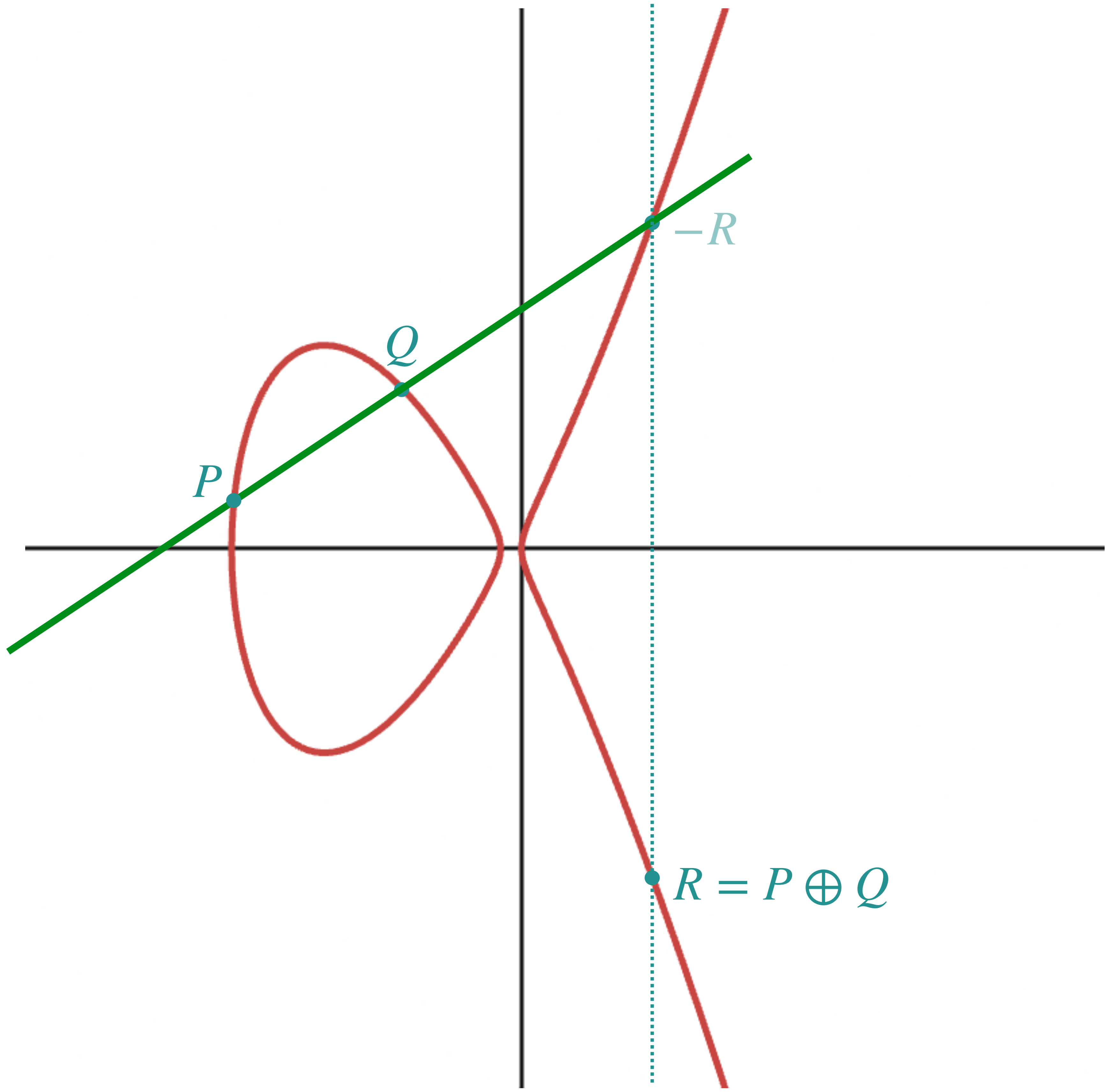
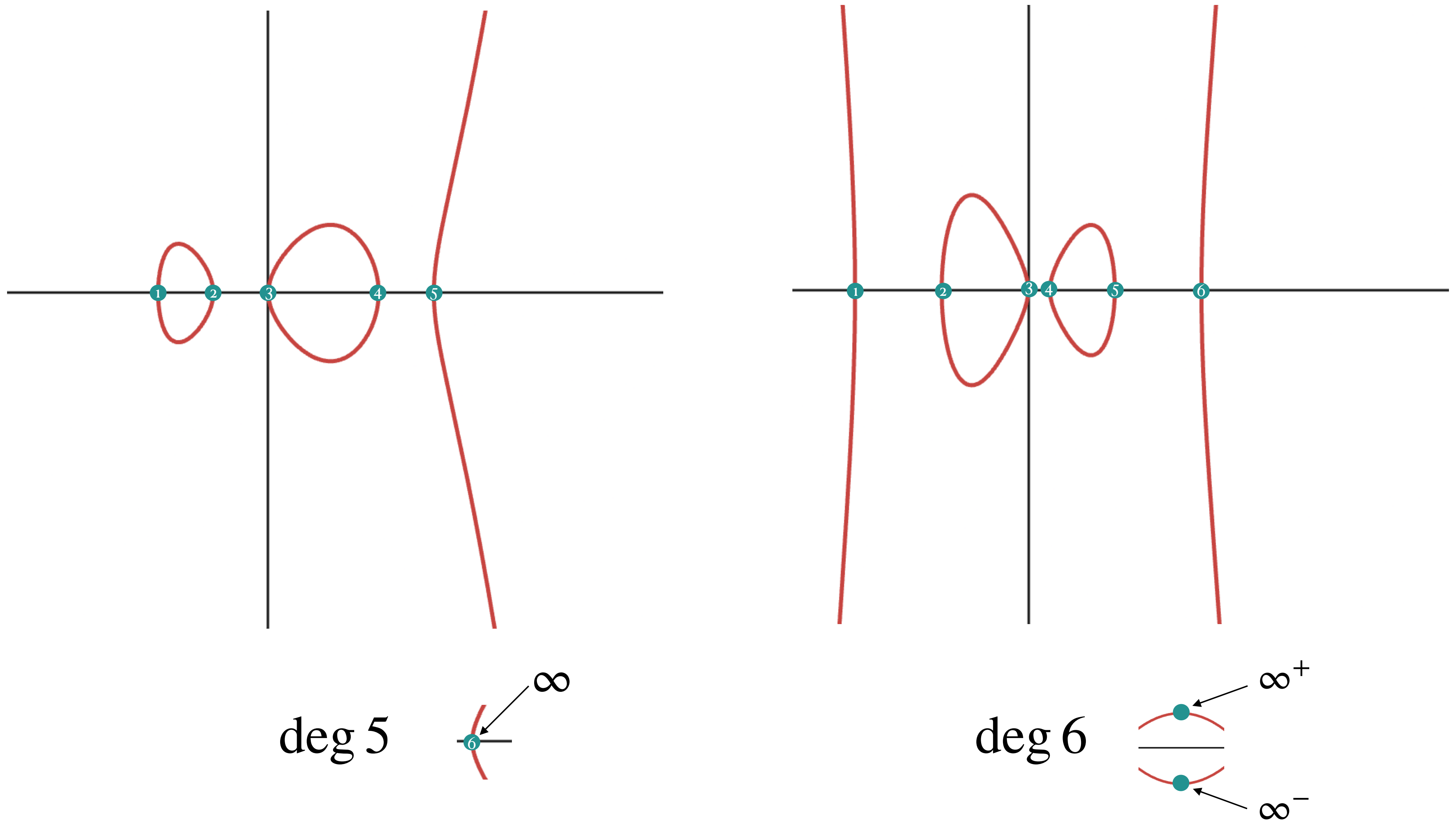


Genus-2 Hyperelliptic Curve

$$C\colon y^2 = f(x), \quad \deg(f) = 5 \text{ or } 6$$



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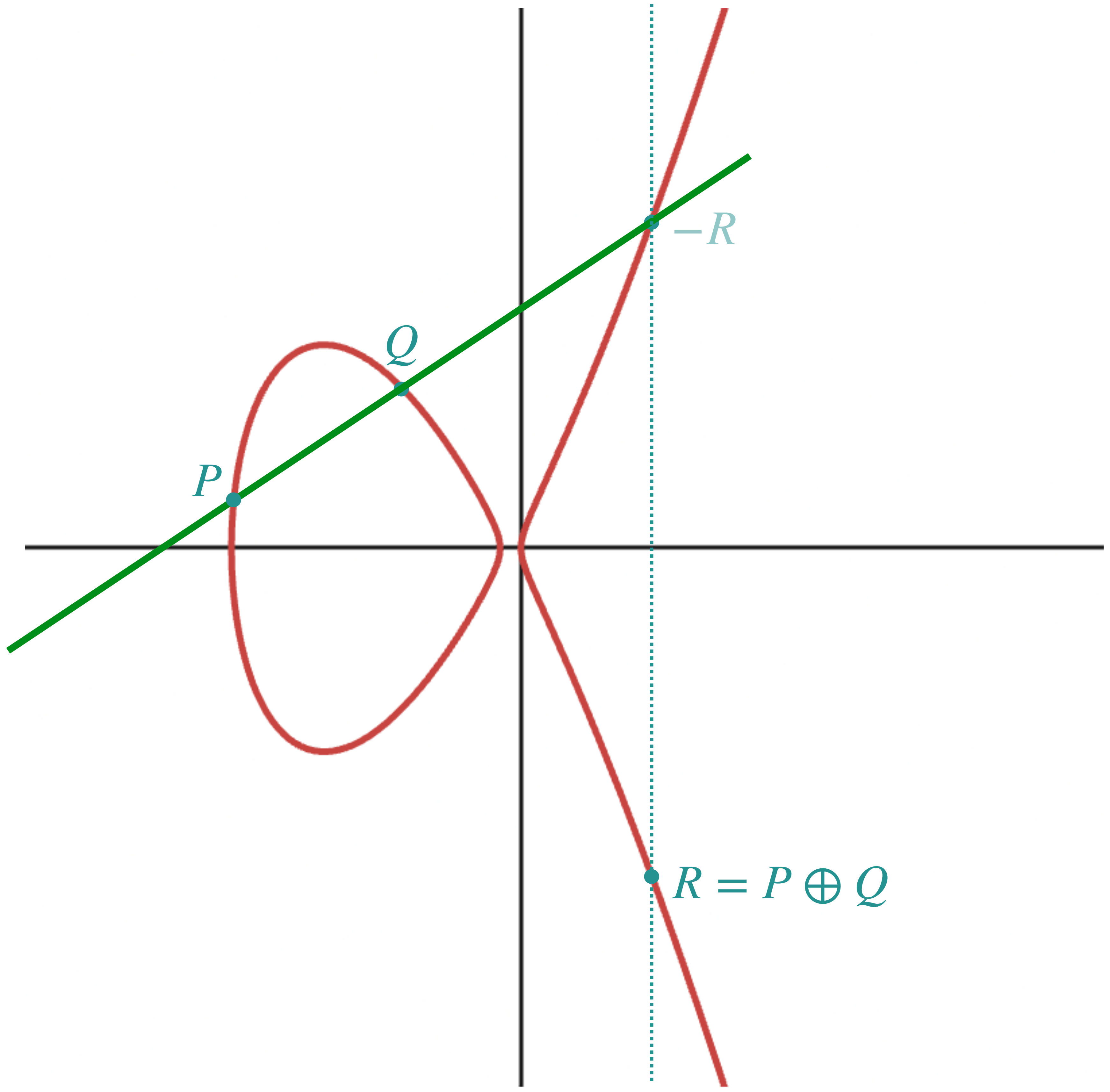

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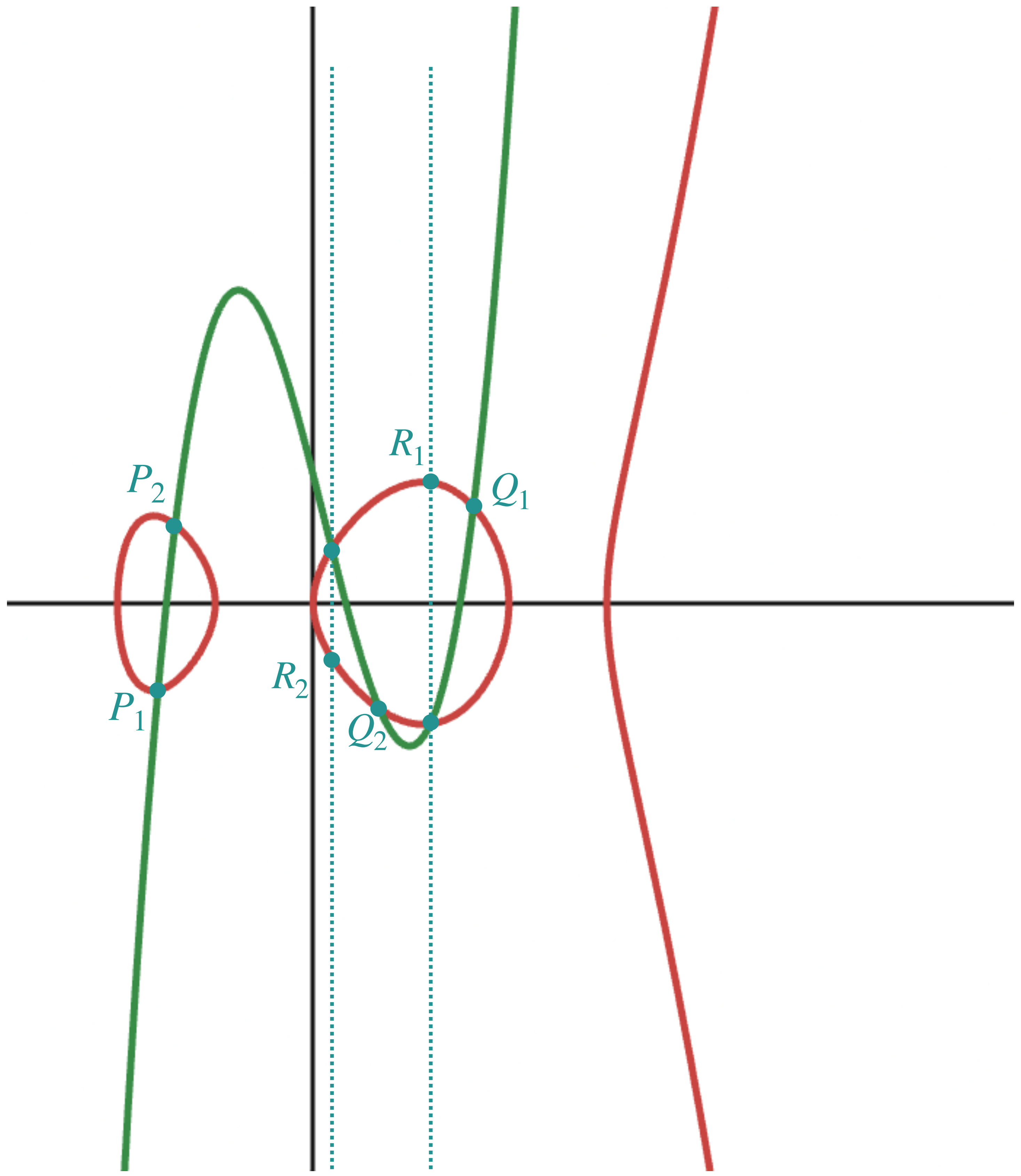

Moving to higher dimension

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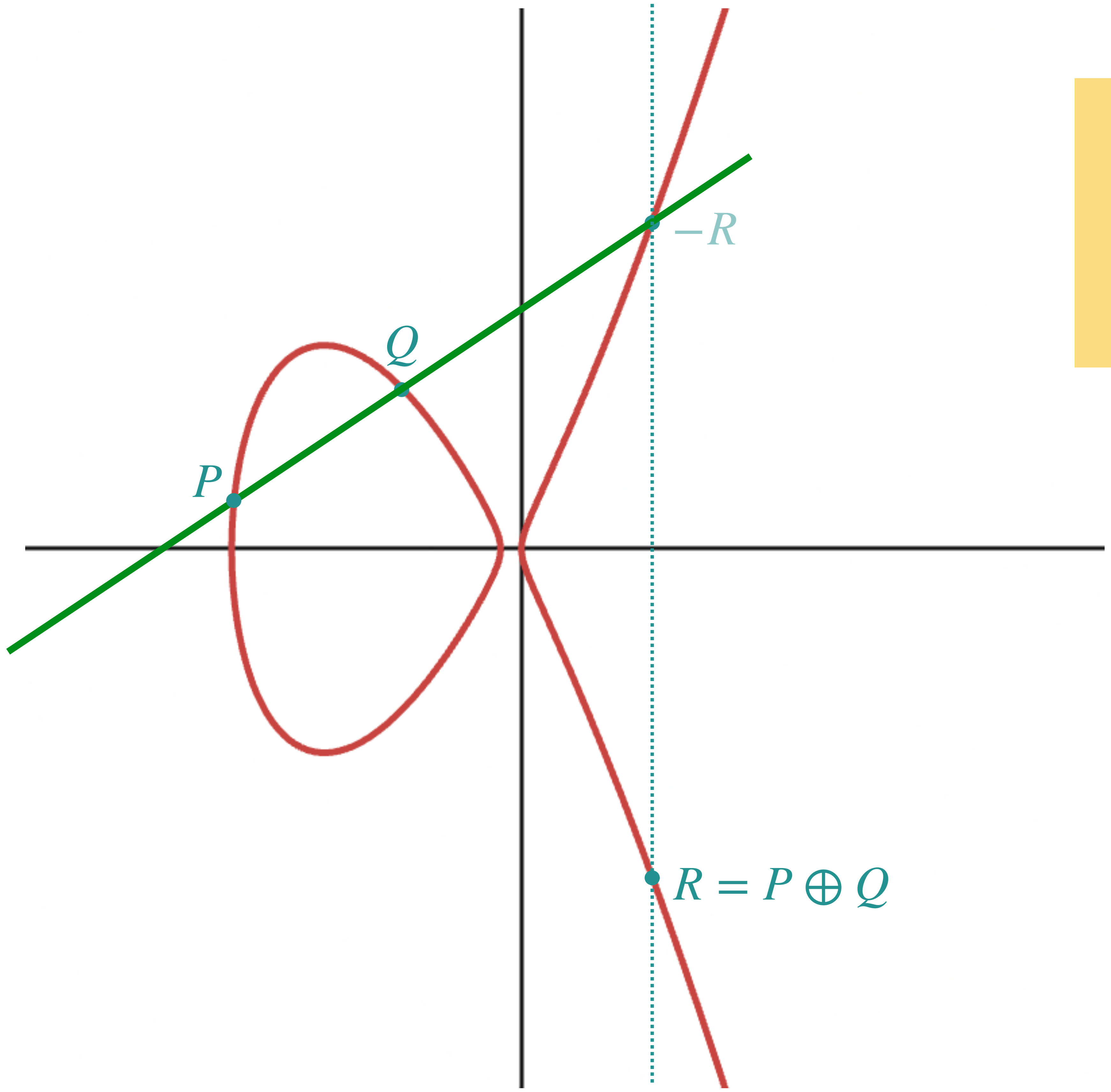
$E: y^2 = x^3 + Ax + B$



The Jacobian is the group attached to C.

In genus 1, $J_C \cong C$.

Genus-2 Hyperelliptic Curve

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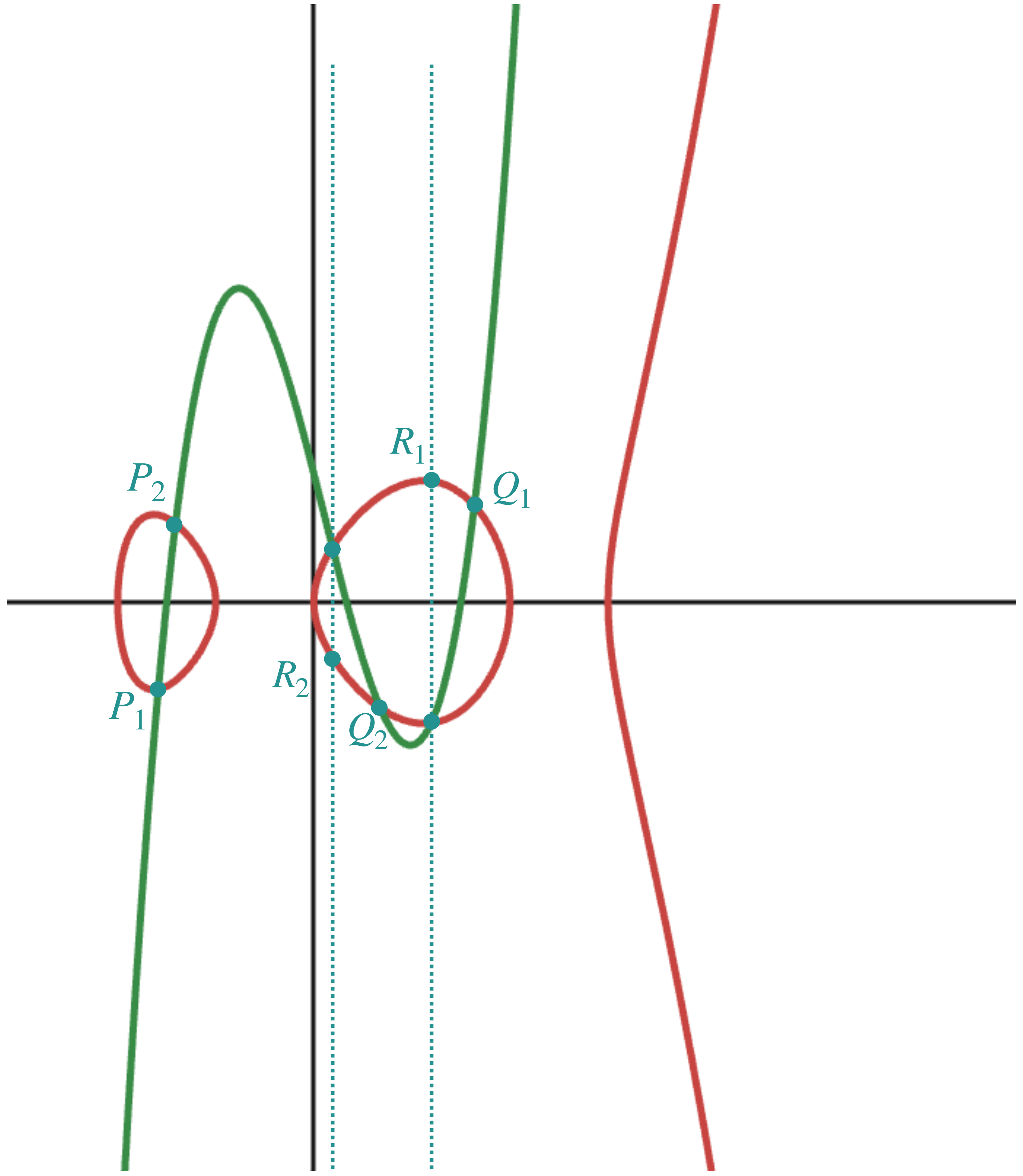

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We now consider **superspecial** (principally polarised) **abelian surfaces** (A, λ).

There are two types:

- Products of elliptic curves $E_1 \times E_2$
- Jacobians J_C of genus 2 hyperelliptic curves C

Definition

An isogeny $\Phi : A_1 \to A_2$ between PPAS is a surjective group homomorphism with finite kernel.

We say it is a (polarised) ℓ-isogeny if:

- $\ker(\Phi) \cong (\mathbb{Z}/\ell\mathbb{Z})^2$
- The isogeny respects the polarisations

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We say it is a (polarised) ℓ-isogeny if:

- $\ker(\Phi) = \langle R, S \rangle$ where $R, S \in A[\ell]$ and $e_\ell(R, S) = 1$

Types of isogenies

Gluing

$$E_1 \times E_2 \to J_C$$

Generic

$$J_1 \to J_2$$

Splitting

$$J_C \to E_1 \times E_2$$

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gluing 2-isogeny

$$\underbrace{E_1 \times E_2 \to J_1}_{\text{gluing 2-isogeny}} \to \ldots \to J_k \to E_3 \times E_4$$

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generic 2-isogenies

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splitting 2-isogeny

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$$E_1 \times E_2 \to J_1 \to \ldots \to J_k \to E_3 \times E_4$$

Let's focus on this for now

III. Kummer Surfaces

Motivation: the Kummer line

Models of Elliptic Curves

Short Weierstrass

$$E\colon y^2 = x^3 + Ax + B$$

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If we have rational
2-torsion on E $\cong$

Montgomery Form

$$E: y^2 = x(x - \alpha)(x - 1/\alpha)$$

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$$E\colon y^2 = x(x-\alpha)(x-1/\alpha)$$

$$\xrightarrow{\pi}$$

$$\pi : P \mapsto \begin{cases} (1:0) \text{ if } P = \mathcal{O} \\ (x:1) \text{ if } P = (x,y) \end{cases}$$

Montgomery Kummer Line

$$E/\langle \pm 1 \rangle \hookrightarrow \mathbb{P}^1$$

Many Kummer Line models

The Montgomery Model $(X : Z)$

$$\mathcal{O} = (1 : 0),\ T_1 = (0 : 1),$$
$$T_2 = (a : b),\ T_3 = (b : a)\,.$$

The Theta Model $(\theta_0 : \theta_1)$

$$\mathcal{O} = (a : b),\ T_1 = (-a : b),$$
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$$(X : Z) \mapsto (a(X - Z) : b(X + Z))$$

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$$(\theta_0 : \theta_1) \mapsto (a\theta_1 + b\theta_0 : a\theta_1 - b\theta_0)$$

Kummer Surfaces

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$$C_{\lambda,\mu,\nu}: y^2 = x(x-1)(x-\lambda)(x-\mu)(x-\nu)$$

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$\xrightarrow{\pi : J_C \to J_C/\langle \pm 1 \rangle \hookrightarrow \mathbb{P}^3}$

Cassels—Flynn Kummer surfaces

Cassels and Flynn. *Prolegomena to a Middlebrow Arithmetic of Curves of Genus 2* (1996).

Kummer Surfaces

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Kummer surfaces arising from theta functions of level 2

Gaudry. *Fast genus 2 arithmetic based on Theta functions* (2007).

The theta model

The Kummer surface can be embedded into $\mathbb{P}^3$ with theta coordinates. The system of theta coordinates (of level 2) is fully determined by the *theta null point* $\mathcal{O} = (a : b : c : d)$.

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Two-torsion points

$$\begin{array}{cccc}
(a : b : c : d) & (a : -b : c : -d) & (a : b : -c : -d) & (a - : b : -c : d) \\
(b : a : d : c) & (b : -a : d : -c) & (b : a : -d : -c) & (b : -a : -d : c) \\
(c : d : a : b) & (c : -d : a : -b) & (c : d : -a : -b) & (c : -d : -a : b) \\
(d : c : b : a) & (d : -c : b : -a) & (d : c : -b : -a) & (d : -c : -b : a)
\end{array}$$

Symmetry in the 2-torsion gives us fast doubling and 2-isogenies!

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Two-torsion points

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(b : a : d : c) & (b : -a : d : -c) & (b : a : -d : -c) & (b : -a : -d : c) \\
(c : d : a : b) & (c : -d : a : -b) & (c : d : -a : -b) & (c : -d : -a : b) \\
(d : c : b : a) & (d : -c : b : -a) & (d : c : -b : -a) & (d : -c : -b : a)
\end{array}
$$

There are different systems of theta coordinates and appropriate change of coordinates formulas; we fix a specific choice.

The theta model: arithmetic

The group law is destroyed by quotienting by $\langle \pm 1 \rangle$ but we still have a *pseudo-group law*.

Fast arithmetic : built from simple building blocks

$$\mathrm{H} : (X_1 : X_2 : X_3 : X_4) \mapsto (X_1 + X_2 + X_3 + X_4 : X_1 + X_2 - X_3 - X_4 : X_1 - X_2 + X_3 - X_4 : X_1 - X_2 - X_3 + X_4)$$

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$$\mathrm{S} : (X_1 : X_2 : X_3 : X_4) \mapsto (X_1^2 : X_2^2 : X_3^2 : X_4^2)$$

$$\mathrm{C}_U : (X_1 : X_2 : X_3 : X_4) \mapsto (X_1 \cdot U_1 : X_2 \cdot U_2 : X_3 \cdot U_3 : X_4 \cdot U_4)$$

IV. Computing Generic Isogenies

Dartois, Maino, Pope and Robert. *An Algorithmic Approach to (2,2)-isogenies in the Theta Model* (2024).

The theta model: isogenies

A 2-isogeny $J \to J'$ induces an algebraic map between Kummer surfaces.

$$
\begin{array}{ccc}
J & \longrightarrow & J' \\
\downarrow & & \downarrow \\
J/\langle \pm 1 \rangle & \xrightarrow{\Phi} & J'/\langle \pm 1 \rangle
\end{array}
$$

The theta model: isogenies

A 2-isogeny $J \to J'$ induces an algebraic map between Kummer surfaces.

$$\begin{tikzcd} J \arrow[r] \arrow[d] & J' \arrow[d] \\ J/\langle \pm 1 \rangle \arrow[r, "\Phi"] & J'/\langle \pm 1 \rangle \end{tikzcd}$$

“We lose nothing by going down to the Kummers” - Cassels and Flynn

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A 2-isogeny $J \to J'$ induces an algebraic map between Kummer surfaces.

$$
\begin{tikzcd}
J \arrow[r] \arrow[d] & J' \arrow[d] \\
J/\langle \pm 1 \rangle \arrow[r, "\Phi"] & J'/\langle \pm 1 \rangle
\end{tikzcd}
$$

“Computing (generic) isogenies between abelian surfaces with theta coordinates of level 2”

The theta model: isogenies

Let A, B be abelian surfaces in (a specific choice of) level-2 theta coordinates.

Key Fact:

Let $\Phi : A \to B$ be a 2-isogeny where $\mathcal{O}_A = (a : b : c : d)$. Then

$$\mathcal{O}_B = \mathsf{H} \circ \mathsf{S}^{-1} \circ \mathsf{H} \circ \mathsf{S}(\mathcal{O}_A)$$

Computing generic isogenies

Algorithm 8.34 $\mathsf{GenericCodomain}(0_A)$

Input: The theta constants $0_A := (a : b : c : d)$ of A.

Output: Dual isogeneous theta null point $(\alpha : \beta : \gamma : \delta)$, the inverse of the dual isogeneous theta null point $(\alpha^{-1} : \beta^{-1} : \gamma^{-1} : \delta^{-1})$, and the theta null point 0_B on B. // Case $\alpha \cdot \beta \cdot \gamma \cdot \delta \neq 0$.

$(\alpha^2, \beta^2, \gamma^2, \delta^2) \leftarrow \mathcal{H} \circ \mathcal{S}(a, b, c, d)$

$\alpha \leftarrow \alpha^2$

$\beta \leftarrow \alpha^2 \cdot \beta^2$

$\gamma \leftarrow \alpha^2 \cdot \gamma^2$

$\delta \leftarrow \alpha^2 \cdot \delta^2$

$\beta \leftarrow \mathsf{SquareRoot}(\beta)$

$\gamma \leftarrow \mathsf{SquareRoot}(\gamma)$

$\delta \leftarrow \mathsf{SquareRoot}(\delta)$

$\alpha^{-1} \leftarrow \gamma^2 \cdot \delta^2$

$\beta^{-1} \leftarrow \alpha^{-1} \cdot \beta$

$\alpha^{-1} \leftarrow \alpha^{-1} \cdot \beta^2$

$\gamma^{-1} \leftarrow \delta^2 \cdot \beta^2 \cdot \gamma$

$\delta^{-1} \leftarrow \gamma^2 \cdot \beta^2 \cdot \delta$

$(a_2, b_2, c_2, d_2) \leftarrow \mathcal{H}(\alpha, \beta, \gamma, \delta)$

$0_B \leftarrow (a_2, b_2, c_2, d_2)$

return $(\alpha, \beta, \gamma, \delta), (\alpha^{-1}, \beta^{-1}, \gamma^{-1}, \delta^{-1}), 0_B$ // Total cost: $4\mathbf{S} + 10\mathbf{M} + 24\mathbf{a} + 3\mathbf{Sqrt}$

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Otherwise, A would correspond to a product of elliptic curves

$(\alpha^2, \beta^2, \gamma^2, \delta^2) \leftarrow \mathcal{H} \circ \mathcal{S}(a, b, c, d)$
$\alpha \leftarrow \alpha^2$
$\beta \leftarrow \alpha^2 \cdot \beta^2$
$\gamma \leftarrow \alpha^2 \cdot \gamma^2$
$\delta \leftarrow \alpha^2 \cdot \delta^2$
$\beta \leftarrow \mathsf{SquareRoot}(\beta)$
$\gamma \leftarrow \mathsf{SquareRoot}(\gamma)$
$\delta \leftarrow \mathsf{SquareRoot}(\delta)$
$\alpha^{-1} \leftarrow \gamma^2 \cdot \delta^2$
$\beta^{-1} \leftarrow \alpha^{-1} \cdot \beta$
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$\gamma^{-1} \leftarrow \delta^2 \cdot \beta^2 \cdot \gamma$
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$(a_2, b_2, c_2, d_2) \leftarrow \mathcal{H}(\alpha, \beta, \gamma, \delta)$
$0_B \leftarrow (a_2, b_2, c_2, d_2)$
return $(\alpha, \beta, \gamma, \delta), (\alpha^{-1}, \beta^{-1}, \gamma^{-1}, \delta^{-1}), 0_B$ // Total cost: $4\mathbf{S} + 10\mathbf{M} + 24\mathbf{a} + 3\mathbf{Sqrt}$

Computing generic isogenies

Algorithm 8.34 $\mathsf{GenericCodomain}(0_A)$

Input: The theta constants $0_A := (a : b : c : d)$ of A.

Output: Dual isogeneous theta null point $(\alpha : \beta : \gamma : \delta)$, the inverse of the dual isogeneous theta null point $(\alpha^{-1} : \beta^{-1} : \gamma^{-1} : \delta^{-1})$, and the theta null point 0_B on B. // Case $\alpha \cdot \beta \cdot \gamma \cdot \delta \neq 0$.

$(\alpha^2, \beta^2, \gamma^2, \delta^2) \leftarrow \mathcal{H} \circ \mathcal{S}(a, b, c, d)$

$\alpha \leftarrow \alpha^2$

$\beta \leftarrow \alpha^2 \cdot \beta^2$

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$\delta \leftarrow \alpha^2 \cdot \delta^2$

$\beta \leftarrow \mathsf{SquareRoot}(\beta)$

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Computing S^{-1}

$\alpha^{-1} \leftarrow \gamma^2 \cdot \delta^2$

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What we've done so far

Codomain computation

$$E_1 \times E_2 \to A_1 \to \dots \to A_k \to E_3 \times E_4$$

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But square roots are *expensive*…

Removing square roots

In analogy with dimension one, torsion points allow us to avoid square roots!

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Fun Fact

Let R', S' be such that $\ker(\Phi) = \langle [4]R', [4]S' \rangle$. Then:

$$\mathsf{H} \circ \mathsf{S}(R') = (x\alpha : x\beta : y\gamma : y\delta)$$
$$\mathsf{H} \circ \mathsf{S}(S') = (z\alpha : w\beta : z\gamma : w\delta)$$

for some $x, y, z, w \in \mathbb{F}_{p^2}$.

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for some $x, y, z, w \in \mathbb{F}_{p^2}$.

This tells us we can extract $(\alpha : \beta : \gamma : \delta)$ using a few squaring and multiplications.

We can do something similar with 4-torsion, but we need 2 square roots.

Evaluating points

We compute images of points in an analogous way.

Let $\Phi : A \to B$ be a 2-isogeny where $\mathcal{O}_A = (a : b : c : d)$. Then the image of a point P is given by

$$\mathsf{H} \circ \mathsf{C} \circ \mathsf{H} \circ \mathsf{S}(P)$$

where $\mathsf{C} : (x : y : z : w) \mapsto (x \cdot \alpha^{-1} : y \cdot \beta^{-1} : z \cdot \gamma^{-1} : w \cdot \delta^{-1})$.

We have already computed
$(\alpha^{-1} : \beta^{-1} : \gamma^{-1} : \delta^{-1})$

What we've done so far

Codomain computation and evaluation

$$E_1 \times E_2 \to A_1 \to \ldots \to A_k \to E_3 \times E_4$$

Without square roots!

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Codomain computation and evaluation

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Without square roots!

But we need gluings and splittings!

V. Gluing and splitting isogenies

Splitting isogenies

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Generic Isogeny
$\varphi : A \to B$
where $B \cong E_1 \times E_2$

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Product Theta Structure

Let $B = E \times E'$ be a product of Montgomery elliptic curves each with their system of theta coordinates

$$(\theta_0 : \theta_1) \quad \text{and} \quad (\theta'_0 : \theta'_1)$$

The *product theta coordinates* on B are defined as

$$(x : y : z : w) = (\theta_0\theta'_0 : \theta_1\theta'_0 : \theta_0\theta'_1 : \theta_1\theta'_1)$$

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Recover Montgomery coefficients of E_1, E_2 from $\mathcal{O}_{E_1 \times E_2}$.

Recall...

The Montgomery Model $(X : Z)$

$$\mathcal{O} = (1 : 0),\ T_1 = (0 : 1),$$
$$T_2 = (a : b),\ T_3 = (b : a)\,.$$

The Theta Model $(\theta_0 : \theta_1)$

$$\mathcal{O} = (a : b),\ T_1 = (-a : b),$$
$$T_2 = (b : a),\ T_3 = (-b : a)\,.$$

$$(\theta_0 : \theta_1) \mapsto (a\theta_1 + b\theta_0 : a\theta_1 - b\theta_0)$$

Splitting isogenies

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Procedure is similar for points…

Gluing isogenies

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Apply change of basis matrix to move points $(P_1, P_2) \in E_1 \times E_2$ from Montgomery coordinates to (non-product) theta coordinates (corresp. to our choice of theta structure)

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Product to Theta Structure

Let $E \times E'$ be a product of Montgomery elliptic curves each with product theta coordinates

$$(x : y : z : w) = (\theta_0\theta_0' : \theta_1\theta_0' : \theta_0\theta_1' : \theta_1\theta_1')$$

Then apply a change of basis matrix.

Gluing isogenies

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Compute the codomain of the gluing isogeny

$$E_1 \times E_2 \to A$$

(taking into account we always have $\delta = 0$).

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Compute the codomain of the gluing isogeny

$$E_1 \times E_2 \to A$$

(taking into account we always have $\delta = 0$).

Procedure is similar for evaluating points (P_1, P_2)…
Special (faster) evaluation for points of the form $(P_1,0)$ or $(0,P_2)$

VI. Open Questions

Open Questions

(thanks also to Luciano and Pierrick!)

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How can we best treat cases when a splitting (and therefore then a gluing) happens in the middle of the chain? How can we treat cases where gluing does not happen on the first step?

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? Improve gluing! (coming up next…)

Resources (besides the spec)

An Algorithmic Approach to (2,2)-isogenies in the Theta Model

Dartois, Maino, Pope and Robert

Some notes on algorithms for abelian varieties

Robert

Isogenies on Kummer Surfaces

Joint with Flynn