Quaternionic multiplication and abelian fourfolds

Enric Florit

Universitat de Barcelona

The SQIparty 2025

- Which quaternion algebras can act on a given abelian variety?

- Which quaternion algebras can act on a given abelian variety?
- If a quaternion algebra acts on an abelian variety, what properties does it force on it?

- Which quaternion algebras can act on a given abelian variety?
  $E/\mathbb{F}_q$: only $D_{p,\infty}$

- If a quaternion algebra acts on an abelian variety, what properties does it force on it?

- Which quaternion algebras can act on a given abelian variety?
  $E/\mathbb{F}_q$: only $D_{p,\infty}$

- If a quaternion algebra acts on an abelian variety, what properties does it force on it?
  $E/\mathbb{F}_q$: E supersingular, $E[p] = 0$.

A local-global problem

Problem A

- k number field, A/k simple abelian variety, Σ_A =primes of good red.
- $\mathfrak{p} \in \Sigma_A \rightsquigarrow A_{\mathfrak{p}} := A \mod \mathfrak{p}$.
- Say $A_{\mathfrak{p}}$ **splits** if $A_{\mathfrak{p}} \sim A_1 \times A_2$; **simple** otherwise.

A local-global problem

Problem A

- k number field, A/k simple abelian variety, Σ_A =primes of good red.
- $\mathfrak{p} \in \Sigma_A \rightsquigarrow A_{\mathfrak{p}} := A \mod \mathfrak{p}$.
- Say $A_{\mathfrak{p}}$ **splits** if $A_{\mathfrak{p}} \sim A_1 \times A_2$; **simple** otherwise.
- Suppose $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$ is non-commutative.

A local-global problem

Problem A

- k number field, A/k simple abelian variety, Σ_A =primes of good red.
- $\mathfrak{p} \in \Sigma_A \rightsquigarrow A_\mathfrak{p} := A \mod \mathfrak{p}$.
- Say $A_\mathfrak{p}$ **splits** if $A_\mathfrak{p} \sim A_1 \times A_2$; **simple** otherwise.
- Suppose $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$ is non-commutative.
- **Murty-Patankar Problem:** characterize the set

$$S = \{\mathfrak{p} \in \Sigma_A \colon A_\mathfrak{p} \text{ is simple}\}$$

A local-global problem

Problem A

- k number field, A/k simple abelian variety, Σ_A =primes of good red.
- $\mathfrak{p} \in \Sigma_A \rightsquigarrow A_\mathfrak{p} := A \mod \mathfrak{p}$.
- Say $A_\mathfrak{p}$ **splits** if $A_\mathfrak{p} \sim A_1 \times A_2$; **simple** otherwise.
- Suppose $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$ is non-commutative.
- **Murty-Patankar Problem:** characterize the set

$$S = \{\mathfrak{p} \in \Sigma_A \colon A_\mathfrak{p} \text{ is simple}\}$$

For $\mathfrak{p} \in \Sigma_A$, there is a "reduction embedding"

$$\mathrm{End}^0(A) \to \mathrm{End}^0(A_\mathfrak{p}).$$

We will relate the two algebras and study whether $\mathrm{End}^0(A_\mathfrak{p})$ is division or contains zero-divisors.

A classical theorem

Theorem (Morita, Yoshida, 70s)

Let A/k be a simple abelian surface with $D = \mathrm{End}^0(A)$ indefinite quaternion algebra.

A classical theorem

Theorem (Morita, Yoshida, 70s)

Let A/k be a simple abelian surface with $D = \mathrm{End}^0(A)$ indefinite quaternion algebra.
If $A_{\mathfrak{p}}$ is simple, then D ramifies at p.

A classical theorem

Theorem (Morita, Yoshida, 70s)

Let A/k be a simple abelian surface with $D = \mathrm{End}^0(A)$ indefinite quaternion algebra.
If $A_{\mathfrak{p}}$ is simple, then D ramifies at p.
$\rightsquigarrow S = \{\mathfrak{p} \in \Sigma_A \colon A_{\mathfrak{p}} \text{ is simple}\}$ is **finite**.

A classical theorem

Theorem (Morita, Yoshida, 70s)

Let A/k be a simple abelian surface with $D = \mathrm{End}^0(A)$ indefinite quaternion algebra.
If $A_\mathfrak{p}$ is simple, then D ramifies at p.
$\rightsquigarrow S = \{\mathfrak{p} \in \Sigma_A \colon A_\mathfrak{p} \text{ is simple}\}$ is **finite**.

Proof. Let π be Frobenius on $A_\mathfrak{p}$. We reason by cases on $\mathbb{Q}(\pi)$.

A classical theorem

Theorem (Morita, Yoshida, 70s)

Let A/k be a simple abelian surface with $D = \mathrm{End}^0(A)$ indefinite quaternion algebra.
If $A_\mathfrak{p}$ is simple, then D ramifies at p.
$\rightsquigarrow S = \{\mathfrak{p} \in \Sigma_A \colon A_\mathfrak{p} \text{ is simple}\}$ is **finite**.

Proof. Let π be Frobenius on $A_\mathfrak{p}$. We reason by cases on $\mathbb{Q}(\pi)$.

- $\mathbb{Q}(\pi) = \mathbb{Q} \implies A \sim E^2$, supersingular.

A classical theorem

Theorem (Morita, Yoshida, 70s)

Let A/k be a simple abelian surface with $D = \mathrm{End}^0(A)$ indefinite quaternion algebra.
If $A_{\mathfrak{p}}$ is simple, then D ramifies at p.
$\rightsquigarrow S = \{\mathfrak{p} \in \Sigma_A \colon A_{\mathfrak{p}} \text{ is simple}\}$ is **finite**.

Proof. Let π be Frobenius on $A_{\mathfrak{p}}$. We reason by cases on $\mathbb{Q}(\pi)$.

- $\mathbb{Q}(\pi) = \mathbb{Q} \implies A \sim E^2$, supersingular.
- $\mathbb{Q}(\pi) =$imag. quadr., can show

$$D \otimes_{\mathbb{Q}} \mathbb{Q}(\pi) \simeq \mathrm{End}^0(A_{\mathfrak{p}}).$$

A classical theorem

Theorem (Morita, Yoshida, 70s)

Let A/k be a simple abelian surface with $D = \mathrm{End}^0(A)$ indefinite quaternion algebra.
If $A_{\mathfrak{p}}$ is simple, then D ramifies at p.
$\rightsquigarrow S = \{\mathfrak{p} \in \Sigma_A \colon A_{\mathfrak{p}} \text{ is simple}\}$ is **finite**.

Proof. Let π be Frobenius on $A_{\mathfrak{p}}$. We reason by cases on $\mathbb{Q}(\pi)$.

- $\mathbb{Q}(\pi) = \mathbb{Q} \implies A \sim E^2$, supersingular.
- $\mathbb{Q}(\pi) =$imag. quadr., can show

$$D \otimes_{\mathbb{Q}} \mathbb{Q}(\pi) \simeq \mathrm{End}^0(A_{\mathfrak{p}}).$$

If $A_{\mathfrak{p}}$ simple, then $\mathrm{End}^0(A_{\mathfrak{p}})$ ramifies at $p \implies D$ ramifies at p.

A classical theorem

Theorem (Morita, Yoshida, 70s)

Let A/k be a simple abelian surface with $D = \mathrm{End}^0(A)$ indefinite quaternion algebra.
If $A_{\mathfrak{p}}$ is simple, then D ramifies at p.
$\rightsquigarrow S = \{\mathfrak{p} \in \Sigma_A \colon A_{\mathfrak{p}} \text{ is simple}\}$ is **finite**.

Proof. Let π be Frobenius on $A_{\mathfrak{p}}$. We reason by cases on $\mathbb{Q}(\pi)$.

- $\mathbb{Q}(\pi) = \mathbb{Q} \implies A \sim E^2$, supersingular.
- $\mathbb{Q}(\pi)$ =imag. quadr., can show

$$D \otimes_{\mathbb{Q}} \mathbb{Q}(\pi) \simeq \mathrm{End}^0(A_{\mathfrak{p}}).$$

  If $A_{\mathfrak{p}}$ simple, then $\mathrm{End}^0(A_{\mathfrak{p}})$ ramifies at $p \implies D$ ramifies at p.
- $\mathbb{Q}(\pi) = \mathbb{Q}(\sqrt{p})$ would also give

$$D \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{p}) \simeq \mathrm{End}^0(A_{\mathfrak{p}}).$$

  But D is indefinite, while $\mathrm{End}^0(A_{\mathfrak{p}})$ is definite (Tate).

$\square$

A second problem

In fact, we have almost solved the following problem:

Problem B

Let p prime, $q = p^r$, $B/\mathbb{F}_q$ be an abelian surface.
Let $D/\mathbb{Q}$ an indefinite quaternion algebra.

1. Characterize the existence of an embedding $\iota : D \to \mathrm{End}^0(B)$.
2. If ι exists, determine splitting and p-rank of B.

Classifying surfaces with QM

The proof of Morita–Yoshida gives the solution to Problem B for surfaces.

Theorem (Chia-Fu Yu)

Let $D/\mathbb{Q}$ indefinite division quaternion algebra, $B/\mathbb{F}_q$ abelian surface with embedding $D \to \mathrm{End}^0(B)$. Then either

1. $A \sim E^2$, with
   1. $\mathrm{End}^0(A) \simeq \mathrm{Mat}_2(D_{p,\infty})$, E supersingular, or
   2. $\mathrm{End}^0(A) \simeq \mathrm{Mat}_2(\mathbb{Q}(\pi))$, $\mathbb{Q}(\pi)$ splitting D.
2. A simple, supersingular, with $\mathrm{End}^0(A) \simeq D \otimes \mathbb{Q}(\pi)$, $\mathbb{Q}(\pi)$ imaginary quadratic.

Solving Problem A

Solving Problem A

Theorem (F.)

Let k number field, A/k simple abelian variety such that $\mathrm{End}(A)$ is non-commutative. Let $\Sigma_A \ni \mathfrak{p} \mid p$.

Solving Problem A

Theorem (F.)

Let k number field, A/k simple abelian variety such that $\mathrm{End}(A)$ is non-commutative. Let $\Sigma_A \ni \mathfrak{p} \mid p$.
If $A_{\mathfrak{p}}$ is simple, then $\mathrm{End}^0(A)$ ramifies at a place over p.

Solving Problem A

Theorem (F.)

Let k number field, A/k simple abelian variety such that $\mathrm{End}(A)$ is non-commutative. Let $\Sigma_A \ni \mathfrak{p} \mid p$.
If $A_{\mathfrak{p}}$ is simple, then $\mathrm{End}^0(A)$ ramifies at a place over p.
$\rightsquigarrow S = \{\mathfrak{p} \in \Sigma_A \colon A_{\mathfrak{p}} \text{ simple}\}$ is **finite**.

Proof

Theorem (F.)

If $A_{\mathfrak{p}}$ is simple, then $\mathrm{End}^0(A)$ ramifies at a place over p.

Proof

Theorem (F.)

If $A_{\mathfrak{p}}$ is simple, then $\mathrm{End}^0(A)$ ramifies at a place over p.

- Let $Z = Z(\mathrm{End}^0(A))$, suppose $A_{\mathfrak{p}}$ simple.

Proof

Theorem (F.)

If $A_{\mathfrak{p}}$ is simple, then $\mathrm{End}^0(A)$ ramifies at a place over p.

- Let $Z = Z(\mathrm{End}^0(A))$, suppose $A_{\mathfrak{p}}$ simple.
- We have an embedding $\iota : \mathrm{End}^0(A) \to \mathrm{End}^0(A_{\mathfrak{p}})$.

Proof

Theorem (F.)

If $A_{\mathfrak{p}}$ is simple, then $\mathrm{End}^0(A)$ ramifies at a place over p.

- Let $Z = Z(\mathrm{End}^0(A))$, suppose $A_{\mathfrak{p}}$ simple.
- We have an embedding $\iota : \mathrm{End}^0(A) \to \mathrm{End}^0(A_{\mathfrak{p}})$.
- The subalg. $Z(\pi) \subset \mathrm{End}^0(A_{\mathfrak{p}})$ generated by $\iota(Z)$ and $\mathbb{Q}(\pi)$ is a field.

Proof

Theorem (F.)

If $A_{\mathfrak{p}}$ is simple, then $\mathrm{End}^0(A)$ ramifies at a place over p.

- Let $Z = Z(\mathrm{End}^0(A))$, suppose $A_{\mathfrak{p}}$ simple.
- We have an embedding $\iota : \mathrm{End}^0(A) \to \mathrm{End}^0(A_{\mathfrak{p}})$.
- The subalg. $Z(\pi) \subset \mathrm{End}^0(A_{\mathfrak{p}})$ generated by $\iota(Z)$ and $\mathbb{Q}(\pi)$ is a field.
- The embedding ι can be extended to

$$\tilde{\iota} : \mathrm{End}^0(A) \otimes_Z Z(\pi) \to \mathrm{End}^0(A_{\mathfrak{p}}).$$

Proof

Theorem (F.)

If $A_{\mathfrak{p}}$ is simple, then $\mathrm{End}^0(A)$ ramifies at a place over p.

- Let $Z = Z(\mathrm{End}^0(A))$, suppose $A_{\mathfrak{p}}$ simple.
- We have an embedding $\iota : \mathrm{End}^0(A) \to \mathrm{End}^0(A_{\mathfrak{p}})$.
- The subalg. $Z(\pi) \subset \mathrm{End}^0(A_{\mathfrak{p}})$ generated by $\iota(Z)$ and $\mathbb{Q}(\pi)$ is a field.
- The embedding ι can be extended to

$$\tilde{\iota} : \mathrm{End}^0(A) \otimes_Z Z(\pi) \to \mathrm{End}^0(A_{\mathfrak{p}}).$$

- By the Double Centralizer Theorem, there is some $t > 0$ and a nontrivial equality in $\mathrm{Br}(Z(\pi))$:

$$t[\mathrm{End}^0(A) \otimes_Z Z(\pi)] = t[\mathrm{End}^0(A_{\mathfrak{p}}) \otimes_{\mathbb{Q}(\pi)} Z(\pi)].$$

Proof

Theorem (F.)

If $A_{\mathfrak{p}}$ is simple, then $\mathrm{End}^0(A)$ ramifies at a place over p.

- Let $Z = Z(\mathrm{End}^0(A))$, suppose $A_{\mathfrak{p}}$ simple.
- We have an embedding $\iota : \mathrm{End}^0(A) \to \mathrm{End}^0(A_{\mathfrak{p}})$.
- The subalg. $Z(\pi) \subset \mathrm{End}^0(A_{\mathfrak{p}})$ generated by $\iota(Z)$ and $\mathbb{Q}(\pi)$ is a field.
- The embedding ι can be extended to

$$\tilde{\iota} : \mathrm{End}^0(A) \otimes_Z Z(\pi) \to \mathrm{End}^0(A_{\mathfrak{p}}).$$

- By the Double Centralizer Theorem, there is some $t > 0$ and a nontrivial equality in $\mathrm{Br}(Z(\pi))$:

$$t[\mathrm{End}^0(A) \otimes_Z Z(\pi)] = t[\mathrm{End}^0(A_{\mathfrak{p}}) \otimes_{\mathbb{Q}(\pi)} Z(\pi)].$$

- Can assume $\dim A \geq 3$, so $\mathbb{Q}(\pi)$ CM field

Proof

Theorem (F.)

If $A_{\mathfrak{p}}$ is simple, then $\mathrm{End}^0(A)$ ramifies at a place over p.

- Let $Z = Z(\mathrm{End}^0(A))$, suppose $A_{\mathfrak{p}}$ simple.
- We have an embedding $\iota : \mathrm{End}^0(A) \to \mathrm{End}^0(A_{\mathfrak{p}})$.
- The subalg. $Z(\pi) \subset \mathrm{End}^0(A_{\mathfrak{p}})$ generated by $\iota(Z)$ and $\mathbb{Q}(\pi)$ is a field.
- The embedding ι can be extended to

$$\tilde{\iota} : \mathrm{End}^0(A) \otimes_Z Z(\pi) \to \mathrm{End}^0(A_{\mathfrak{p}}).$$

- By the Double Centralizer Theorem, there is some $t > 0$ and a nontrivial equality in $\mathrm{Br}(Z(\pi))$:

$$t[\mathrm{End}^0(A) \otimes_Z Z(\pi)] = t[\mathrm{End}^0(A_{\mathfrak{p}}) \otimes_{\mathbb{Q}(\pi)} Z(\pi)].$$

- Can assume $\dim A \geq 3$, so $\mathbb{Q}(\pi)$ CM field $\implies \mathrm{End}^0(A_{\mathfrak{p}})$ ramifies at p

Proof

Theorem (F.)

If $A_{\mathfrak{p}}$ is simple, then $\mathrm{End}^0(A)$ ramifies at a place over p.

- Let $Z = Z(\mathrm{End}^0(A))$, suppose $A_{\mathfrak{p}}$ simple.
- We have an embedding $\iota : \mathrm{End}^0(A) \to \mathrm{End}^0(A_{\mathfrak{p}})$.
- The subalg. $Z(\pi) \subset \mathrm{End}^0(A_{\mathfrak{p}})$ generated by $\iota(Z)$ and $\mathbb{Q}(\pi)$ is a field.
- The embedding ι can be extended to

$$\tilde{\iota} : \mathrm{End}^0(A) \otimes_Z Z(\pi) \to \mathrm{End}^0(A_{\mathfrak{p}}).$$

- By the Double Centralizer Theorem, there is some $t > 0$ and a nontrivial equality in $\mathrm{Br}(Z(\pi))$:

$$t[\mathrm{End}^0(A) \otimes_Z Z(\pi)] = t[\mathrm{End}^0(A_{\mathfrak{p}}) \otimes_{\mathbb{Q}(\pi)} Z(\pi)].$$

- Can assume $\dim A \geq 3$, so $\mathbb{Q}(\pi)$ CM field $\implies \mathrm{End}^0(A_{\mathfrak{p}})$ ramifies at $p \implies \mathrm{End}^0(A)$ ramifies at p. $\square$

Problem A'

Theorem (F.)

Let k number field, A/k simple abelian variety such that $\mathrm{End}(A)$ is non-commutative.
Then, A splits modulo all but finitely many primes.

Problem A'

Theorem (F.)

Let k number field, A/k simple abelian variety such that $\mathrm{End}(A)$ is non-commutative.
Then, A splits modulo all but finitely many primes.

Problem A'

- Can we say anything more about the set of simple reductions?
- Is it ever nonempty?

Some technicalities

Embeddigs of division algebras

The Double Centralizer step told us: $\exists t > 0$,

$$t[\mathrm{End}^0(A) \otimes_Z Z(\pi)] = t[\mathrm{End}^0(A_{\mathfrak{p}}) \otimes_{\mathbb{Q}(\pi)} Z(\pi)].$$

Embeddigs of division algebras

The Double Centralizer step told us: $\exists t > 0$,

$$t[\mathrm{End}^0(A) \otimes_Z Z(\pi)] = t[\mathrm{End}^0(A_\mathfrak{p}) \otimes_{\mathbb{Q}(\pi)} Z(\pi)].$$

The converse also holds.

Theorem

Let Z_Y number field, Y/Z_Y division algebra, F a subfield of Y, D/F be a division quaternion algebra.

Embeddigs of division algebras

The Double Centralizer step told us: $\exists t > 0$,

$$t[\mathrm{End}^0(A) \otimes_Z Z(\pi)] = t[\mathrm{End}^0(A_{\mathfrak{p}}) \otimes_{\mathbb{Q}(\pi)} Z(\pi)].$$

The converse also holds.

Theorem

Let Z_Y number field, Y/Z_Y division algebra, F a subfield of Y, D/F be a division quaternion algebra.

Let $\mathcal{Z} = F \cdot Z_Y \subset Y$, $d := \frac{\mathrm{ord}_{Z_Y}[Y]}{[\mathcal{Z}:Z_Y]} = \frac{\sqrt{\dim_{Z_Y} Y}}{[\mathcal{Z}:Z_Y]}$.

Embeddigs of division algebras

The Double Centralizer step told us: $\exists t > 0$,

$$t[\mathrm{End}^0(A) \otimes_Z Z(\pi)] = t[\mathrm{End}^0(A_{\mathfrak{p}}) \otimes_{\mathbb{Q}(\pi)} Z(\pi)].$$

The converse also holds.

Theorem

Let Z_Y number field, Y/Z_Y division algebra, F a subfield of Y, D/F be a division quaternion algebra.

Let $\mathcal{Z} = F \cdot Z_Y \subset Y$, $d := \frac{\mathrm{ord}_{Z_Y}[Y]}{[\mathcal{Z}:Z_Y]} = \frac{\sqrt{\dim_{Z_Y} Y}}{[\mathcal{Z}:Z_Y]}$.

There exists an embedding $\iota : D \to Y$ if and only if

Embeddigs of division algebras

The Double Centralizer step told us: $\exists t > 0$,

$$t[\mathrm{End}^0(A) \otimes_Z Z(\pi)] = t[\mathrm{End}^0(A_{\mathfrak{p}}) \otimes_{\mathbb{Q}(\pi)} Z(\pi)].$$

The converse also holds.

Theorem

Let Z_Y number field, Y/Z_Y division algebra, F a subfield of Y, D/F be a division quaternion algebra.

Let $\mathcal{Z} = F \cdot Z_Y \subset Y$, $d := \frac{\mathrm{ord}_{Z_Y}[Y]}{[\mathcal{Z}:Z_Y]} = \frac{\sqrt{\dim_{Z_Y} Y}}{[\mathcal{Z}:Z_Y]}$.

There exists an embedding $\iota : D \to Y$ if and only if

1. d is divisible by 2 exactly once.

Embeddigs of division algebras

The Double Centralizer step told us: $\exists t > 0$,

$$t[\mathrm{End}^0(A) \otimes_Z Z(\pi)] = t[\mathrm{End}^0(A_{\mathfrak{p}}) \otimes_{\mathbb{Q}(\pi)} Z(\pi)].$$

The converse also holds.

Theorem

Let Z_Y number field, Y/Z_Y division algebra, F a subfield of Y, D/F be a division quaternion algebra.

Let $\mathcal{Z} = F \cdot Z_Y \subset Y$, $d := \frac{\mathrm{ord}_{Z_Y}[Y]}{[\mathcal{Z}:Z_Y]} = \frac{\sqrt{\dim_{Z_Y} Y}}{[\mathcal{Z}:Z_Y]}$.

There exists an embedding $\iota : D \to Y$ if and only if

1. d is divisible by 2 exactly once.
2. $\frac{d}{2}[D \otimes_F \mathcal{Z}] = \frac{d}{2}[Y \otimes_{Z_Y} \mathcal{Z}]$ in $\mathrm{Br}(\mathcal{Z})$.

Embeddigs of division algebras

The Double Centralizer step told us: $\exists t > 0$,

$$t[\mathrm{End}^0(A) \otimes_Z Z(\pi)] = t[\mathrm{End}^0(A_\mathfrak{p}) \otimes_{\mathbb{Q}(\pi)} Z(\pi)].$$

The converse also holds.

Theorem

Let Z_Y number field, Y/Z_Y division algebra, F a subfield of Y, D/F be a division quaternion algebra.

Let $\mathcal{Z} = F \cdot Z_Y \subset Y$, $d := \frac{\mathrm{ord}_{Z_Y}[Y]}{[\mathcal{Z}:Z_Y]} = \frac{\sqrt{\dim_{Z_Y} Y}}{[\mathcal{Z}:Z_Y]}$.

There exists an embedding $\iota : D \to Y$ if and only if

1. d is divisible by 2 exactly once.
2. $\frac{d}{2}[D \otimes_F \mathcal{Z}] = \frac{d}{2}[Y \otimes_{Z_Y} \mathcal{Z}]$ in $\mathrm{Br}(\mathcal{Z})$.

When the conditions hold, $\mathcal{Z}$ does not split D or Y.

Matrix algebras

- Suppose we have an embedding of simple algebras $X \to Y$.

Matrix algebras

- Suppose we have an embedding of simple algebras $X \to Y$.
- If Y is division, $\mathcal{Z} = Z_X \cdot Z_Y$ is a field,
  and $X \otimes_{Z_X} \mathcal{Z} \to Y$ is an **injection**
  $\rightsquigarrow$ can apply Double Centralizer Theorem.

Matrix algebras

- Suppose we have an embedding of simple algebras $X \to Y$.
- If Y is division, $\mathcal{Z} = Z_X \cdot Z_Y$ is a field,
  and $X \otimes_{Z_X} \mathcal{Z} \to Y$ is an **injection**
  $\rightsquigarrow$ can apply Double Centralizer Theorem.
- In general, $Y \simeq \mathrm{Mat}_r(Y')$ with Y' division.

Matrix algebras

- Suppose we have an embedding of simple algebras $X \to Y$.
- If Y is division, $\mathcal{Z} = Z_X \cdot Z_Y$ is a field,
  and $X \otimes_{Z_X} \mathcal{Z} \to Y$ is an **injection**
  $\rightsquigarrow$ can apply Double Centralizer Theorem.
- In general, $Y \simeq \mathrm{Mat}_r(Y')$ with Y' division.
- We have $X \otimes_{Z_X} \mathcal{Z} \to Y$, but

Matrix algebras

- Suppose we have an embedding of simple algebras $X \to Y$.
- If Y is division, $\mathcal{Z} = Z_X \cdot Z_Y$ is a field,
  and $X \otimes_{Z_X} \mathcal{Z} \to Y$ is an **injection**
  $\rightsquigarrow$ can apply Double Centralizer Theorem.
- In general, $Y \simeq \mathrm{Mat}_r(Y')$ with Y' division.
- We have $X \otimes_{Z_X} \mathcal{Z} \to Y$, but
  1. $\mathcal{Z}$ might not be a field.

Matrix algebras

- Suppose we have an embedding of simple algebras $X \to Y$.
- If Y is division, $\mathcal{Z} = Z_X \cdot Z_Y$ is a field, and $X \otimes_{Z_X} \mathcal{Z} \to Y$ is an **injection** $\rightsquigarrow$ can apply Double Centralizer Theorem.
- In general, $Y \simeq \mathrm{Mat}_r(Y')$ with Y' division.
- We have $X \otimes_{Z_X} \mathcal{Z} \to Y$, but
  1. $\mathcal{Z}$ might not be a field.
  2. $X \otimes_{Z_X} \mathcal{Z} \to Y$ might not be an injection (no DCT!)

Matrix algebras

- Suppose we have an embedding of simple algebras $X \to Y$.
- If Y is division, $\mathcal{Z} = Z_X \cdot Z_Y$ is a field, and $X \otimes_{Z_X} \mathcal{Z} \to Y$ is an **injection** $\rightsquigarrow$ can apply Double Centralizer Theorem.
- In general, $Y \simeq \mathrm{Mat}_r(Y')$ with Y' division.
- We have $X \otimes_{Z_X} \mathcal{Z} \to Y$, but
  1. $\mathcal{Z}$ might not be a field.
  2. $X \otimes_{Z_X} \mathcal{Z} \to Y$ might not be an injection (no DCT!)

Example

Consider the following two embeddings $\mathbb{Q}(\sqrt{5}) \to \mathrm{Mat}_2(\mathbb{Q}(\sqrt{5}))$.

Matrix algebras

- Suppose we have an embedding of simple algebras $X \to Y$.
- If Y is division, $\mathcal{Z} = Z_X \cdot Z_Y$ is a field,
  and $X \otimes_{Z_X} \mathcal{Z} \to Y$ is an **injection**
  $\rightsquigarrow$ can apply Double Centralizer Theorem.
- In general, $Y \simeq \mathrm{Mat}_r(Y')$ with Y' division.
- We have $X \otimes_{Z_X} \mathcal{Z} \to Y$, but
  1. $\mathcal{Z}$ might not be a field.
  2. $X \otimes_{Z_X} \mathcal{Z} \to Y$ might not be an injection (no DCT!)

Example

Consider the following two embeddings $\mathbb{Q}(\sqrt{5}) \to \mathrm{Mat}_2(\mathbb{Q}(\sqrt{5}))$.

- $\sqrt{5} \mapsto \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix}$, $\mathcal{Z} = \mathbb{Q}(\sqrt{5})$.

Matrix algebras

- Suppose we have an embedding of simple algebras $X \to Y$.
- If Y is division, $\mathcal{Z} = Z_X \cdot Z_Y$ is a field, and $X \otimes_{Z_X} \mathcal{Z} \to Y$ is an **injection** $\rightsquigarrow$ can apply Double Centralizer Theorem.
- In general, $Y \simeq \mathrm{Mat}_r(Y')$ with Y' division.
- We have $X \otimes_{Z_X} \mathcal{Z} \to Y$, but
  1. $\mathcal{Z}$ might not be a field.
  2. $X \otimes_{Z_X} \mathcal{Z} \to Y$ might not be an injection (no DCT!)

Example

Consider the following two embeddings $\mathbb{Q}(\sqrt{5}) \to \mathrm{Mat}_2(\mathbb{Q}(\sqrt{5}))$.

- $\sqrt{5} \mapsto \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix}$, $\mathcal{Z} = \mathbb{Q}(\sqrt{5})$.
- $\sqrt{5} \mapsto \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}$ and the compositum $\mathcal{Z} \simeq \mathbb{Q}(\sqrt{5}) \times \mathbb{Q}(\sqrt{5})$.

Primitive embeddings

Definition

Let X/Z_X and Y/Z_Y be simple $\mathbb{Q}$-algebras.
Let $\iota : X \to Y$ be an embedding. We say ι is **primitive** if the subalgebra $\mathcal{Z}$ of Y generated by $\iota(Z_X)$ and Z_Y is a field.

Primitive embeddings

Definition

Let X/Z_X and Y/Z_Y be simple $\mathbb{Q}$-algebras.
Let $\iota : X \to Y$ be an embedding. We say ι is **primitive** if the subalgebra $\mathcal{Z}$ of Y generated by $\iota(Z_X)$ and Z_Y is a field.

Example

Consider the following two embeddings $\mathbb{Q}(\sqrt{5}) \to \mathrm{Mat}_2(\mathbb{Q}(\sqrt{5}))$.

- $\sqrt{5} \mapsto \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix}$ is primitive.
- $\sqrt{5} \mapsto \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}$ is not primitive: $\begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} \sqrt{5} & 0 \\ 0 & -\sqrt{5} \end{pmatrix}$ and the compositum $\mathcal{Z}$ is $\mathbb{Q}(\sqrt{5}) \times \mathbb{Q}(\sqrt{5})$.

Why primitive embeddings?

Lemma

Let X/Z_X simple algebra, Y'/Z_Y division algebra, $\varphi : X \to \mathrm{Mat}_r(Y')$ any embedding.
There exist primitive embeddings $\varphi_i : X \to \mathrm{Mat}_{r_i}(Y')$ with $\sum r_i = r$, such that φ factors (up to conjugation) through

$$\prod \varphi_i : X \to \prod_i \mathrm{Mat}_{r_i}(Y') \subset \mathrm{Mat}_r(Y').$$

Why primitive embeddings?

Lemma

Let X/Z_X simple algebra, Y'/Z_Y division algebra, $\varphi : X \to \mathrm{Mat}_r(Y')$ any embedding.
There exist primitive embeddings $\varphi_i : X \to \mathrm{Mat}_{r_i}(Y')$ with $\sum r_i = r$, such that φ factors (up to conjugation) through

$$\prod \varphi_i : X \to \prod_i \mathrm{Mat}_{r_i}(Y') \subset \mathrm{Mat}_r(Y').$$

$$X \to \begin{pmatrix} * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \end{pmatrix} \subset \mathrm{Mat}_7(Y')$$

Existence of primitive embeddings I

Theorem

Let X/Z_X and Y/Z_Y be simple $\mathbb{Q}$-algebras. There exists a primitive embedding $X \to Y$ if and only if

Existence of primitive embeddings I

Theorem

Let X/Z_X and Y/Z_Y be simple $\mathbb{Q}$-algebras. There exists a primitive embedding $X \to Y$ if and only if

1. There exist embeddings $Z_X \to \bar{\mathbb{Q}}, Z_Y \to \bar{\mathbb{Q}}$ such that the compositum $\mathcal{Z} = Z_X Z_Y$ satisfies

$$[\mathcal{Z} : Z_Y]\sqrt{\dim_{Z_X} X} \mid \sqrt{\dim_{Z_Y} Y}.$$

Existence of primitive embeddings I

Theorem

Let X/Z_X and Y/Z_Y be simple $\mathbb{Q}$-algebras. There exists a primitive embedding $X \to Y$ if and only if

1. There exist embeddings $Z_X \to \bar{\mathbb{Q}}, Z_Y \to \bar{\mathbb{Q}}$ such that the compositum $\mathcal{Z} = Z_X Z_Y$ satisfies

$$[\mathcal{Z} : Z_Y]\sqrt{\dim_{Z_X} X} \mid \sqrt{\dim_{Z_Y} Y}.$$

2. Letting $d = \frac{\sqrt{\dim_{Z_Y} Y}}{[\mathcal{Z}:Z_Y]\sqrt{\dim_{Z_X} X}}$, we have

$$d[X \otimes_{Z_X} \mathcal{Z}] = d[Y \otimes_{Z_Y} \mathcal{Z}] \text{ in } \mathrm{Br}(\mathcal{Z}).$$

*The equality of Brauer classes might be trivial!

Existence of primitive embeddings II

Proposition

Let X/Z_X, Y/Z_Y be $\mathbb{Q}$-algebras. Suppose one of the following holds:

- $Z_X/\mathbb{Q}$ is Galois.
- $Z_Y/\mathbb{Q}$ is Galois.
- $[Z_X : \mathbb{Q}] \leq 4$.
- $[Z_Y : \mathbb{Q}] \leq 4$.

Then, there exists an embedding $\varphi : X \to Y$ if and only if there exists a **primitive** embedding $\psi : X \to Y$.

Problem B for abelian fourfolds

Problem B for abelian fourfolds

Problem B

Let p prime, $q = p^r$, $B \sim (B')^r/\mathbb{F}_q$ be an isotypical abelian fourfold.
Let F be a totally real field, D/F a quaternion algebra.

1. Characterize the existence of an embedding $\iota : D \to \mathrm{End}^0(B)$.
2. If ι exists, determine splitting and p-rank of B.

Problem B for abelian fourfolds

Problem B

Let p prime, $q = p^r$, $B \sim (B')^r/\mathbb{F}_q$ be an isotypical abelian fourfold.
Let F be a totally real field, D/F a quaternion algebra.

1. Characterize the existence of an embedding $\iota : D \to \mathrm{End}^0(B)$.
2. If ι exists, determine splitting and p-rank of B.

- If we want $\iota : D \to \mathrm{End}^0(B)$, then $\mathrm{End}^0(B)$ is non-commutative, and $[\mathbb{Q}(\pi) : \mathbb{Q}]$ is a proper divisor of $2 \dim B = 8$.

Problem B for abelian fourfolds

Problem B

Let p prime, $q = p^r$, $B \sim (B')^r/\mathbb{F}_q$ be an isotypical abelian fourfold.
Let F be a totally real field, D/F a quaternion algebra.

1. Characterize the existence of an embedding $\iota : D \to \mathrm{End}^0(B)$.
2. If ι exists, determine splitting and p-rank of B.

- If we want $\iota : D \to \mathrm{End}^0(B)$, then $\mathrm{End}^0(B)$ is non-commutative, and $[\mathbb{Q}(\pi) : \mathbb{Q}]$ is a proper divisor of $2 \dim B = 8$.
- $\implies [\mathbb{Q}(\pi) : \mathbb{Q}] \leq 4$.

Problem B for abelian fourfolds

Problem B

Let p prime, $q = p^r$, $B \sim (B')^r/\mathbb{F}_q$ be an isotypical abelian fourfold.
Let F be a totally real field, D/F a quaternion algebra.

1. Characterize the existence of an embedding $\iota : D \to \mathrm{End}^0(B)$.
2. If ι exists, determine splitting and p-rank of B.

- If we want $\iota : D \to \mathrm{End}^0(B)$, then $\mathrm{End}^0(B)$ is non-commutative, and $[\mathbb{Q}(\pi) : \mathbb{Q}]$ is a proper divisor of $2 \dim B = 8$.
- $\implies [\mathbb{Q}(\pi) : \mathbb{Q}] \leq 4$.
- There exists an embedding ι if and only if there exists a primitive embedding.

Problem B for abelian fourfolds

Problem B

Let p prime, $q = p^r$, $B \sim (B')^r/\mathbb{F}_q$ be an isotypical abelian fourfold.
Let F be a totally real field, D/F a quaternion algebra.

1. Characterize the existence of an embedding $\iota : D \to \mathrm{End}^0(B)$.
2. If ι exists, determine splitting and p-rank of B.

- If we want $\iota : D \to \mathrm{End}^0(B)$, then $\mathrm{End}^0(B)$ is non-commutative, and $[\mathbb{Q}(\pi) : \mathbb{Q}]$ is a proper divisor of $2 \dim B = 8$.
- $\implies [\mathbb{Q}(\pi) : \mathbb{Q}] \leq 4$.
- There exists an embedding ι if and only if there exists a primitive embedding.
- Can focus on characterizing primitive embeddings!

$\mathbb{Q}(\pi) = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{p})$

Conditions for $D \to \mathrm{End}^0(B)$.

$\mathbb{Q}(\pi) = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{p})$

Conditions for $D \to \mathrm{End}^0(B)$.

B	$\mathbb{Q}(\pi)$	$\iota(F) \cap \mathbb{Q}(\pi)$	D has a real split place?	Conditions
E^4	$\mathbb{Q}$	$\mathbb{Q}$	yes	$[F : \mathbb{Q}] \mid 2$
E^4	$\mathbb{Q}$	$\mathbb{Q}$	no	$[F : \mathbb{Q}] \mid 2$, or $[F : \mathbb{Q}] = 4$ and Br. eq.
S^2	$\mathbb{Q}(\sqrt{p})$	$\mathbb{Q}$	yes	$F = \mathbb{Q}$
S^2	$\mathbb{Q}(\sqrt{p})$	$\mathbb{Q}(\sqrt{p})$	yes	$[F : \mathbb{Q}] \mid 2$
S^2	$\mathbb{Q}(\sqrt{p})$	$\mathbb{Q}$	no	$F = \mathbb{Q}$, or $[F : \mathbb{Q}] = 2$ and Br. eq.
S^2	$\mathbb{Q}(\sqrt{p})$	$\mathbb{Q}(\sqrt{p})$	no	$[F : \mathbb{Q}] \mid 2$, or $[F : \mathbb{Q}] = 4$ and Br. eq.

- E: supersingular elliptic curve.
- S: supersingular abelian surface.
- Br. eq.: $[D \otimes_F F(\pi)] = [\mathrm{End}^0(B) \otimes_{\mathbb{Q}(\pi)} F(\pi)]$.

$\mathbb{Q}(\pi)$ quartic CM field

Proposition (Arai-Takai'23)

Suppose $\mathbb{Q}(\pi)$ is quartic CM. There is an embedding $\iota : D \to \mathrm{End}^0(B)$ if and only if

1. $\mathbb{Q}(\pi)$ contains an isomorphic copy of F, and
2. $\mathrm{End}^0(B) \simeq D \otimes_F \mathbb{Q}(\pi)$.

In particular, B splits if and only if $\mathbb{Q}(\pi)$ splits D.

F and $\mathbb{Q}(\pi)$ quadratic

Theorem

Suppose F is real quadratic and $\mathbb{Q}(\pi)$ is imaginary quadratic. There exists $\iota : D \to \mathrm{End}^0(B)$ if and only if:

1. $B \sim E^4$, with $\mathrm{End}^0(E) = \mathbb{Q}(\pi)$, and $F(\pi)$ splits D.

F and $\mathbb{Q}(\pi)$ quadratic

Theorem

Suppose F is real quadratic and $\mathbb{Q}(\pi)$ is imaginary quadratic. There exists $\iota : D \to \mathrm{End}^0(B)$ if and only if:

1. $B \sim E^4$, with $\mathrm{End}^0(E) = \mathbb{Q}(\pi)$, and $F(\pi)$ splits D.
2. $B \sim S^2$, with S a supersingular $\mathbb{F}_q$-simple abelian surface with $\mathbb{Q}(\pi) = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, $\mathrm{End}^0(S)$ the unique quaternion algebra over $\mathbb{Q}(\pi)$ ramified at the places above p, $[\mathbb{F}_q : \mathbb{F}_p]$ is even, and

$$[D \otimes_F F(\pi)] = [\mathrm{End}^0(S) \otimes_{\mathbb{Q}(\pi)} F(\pi)] \text{ in } \mathrm{Br}(F(\pi)).$$

F and $\mathbb{Q}(\pi)$ quadratic

Theorem

Suppose F is real quadratic and $\mathbb{Q}(\pi)$ is imaginary quadratic. There exists $\iota : D \to \mathrm{End}^0(B)$ if and only if:

1. $B \sim E^4$, with $\mathrm{End}^0(E) = \mathbb{Q}(\pi)$, and $F(\pi)$ splits D.
2. $B \sim S^2$, with S a supersingular $\mathbb{F}_q$-simple abelian surface with $\mathbb{Q}(\pi) = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, $\mathrm{End}^0(S)$ the unique quaternion algebra over $\mathbb{Q}(\pi)$ ramified at the places above p, $[\mathbb{F}_q : \mathbb{F}_p]$ is even, and
$$[D \otimes_F F(\pi)] = [\mathrm{End}^0(S) \otimes_{\mathbb{Q}(\pi)} F(\pi)] \text{ in } \mathrm{Br}(F(\pi)).$$
3. B is geometrically simple, has p-rank 0, and $\mathrm{End}^0(B)$ is a degree-4 division algebra over $\mathbb{Q}(\pi)$, such that
$$[D \otimes_F F(\pi)] = [\mathrm{End}^0(B) \otimes_{\mathbb{Q}(\pi)} F(\pi)] \text{ in } \mathrm{Br}(F(\pi)).$$

In particular, $[\mathbb{F}_q : \mathbb{F}_p]$ is a multiple of 4 and B is not supersingular. The prime p does not split in F, and D ramifies at p.

$F = \mathbb{Q}$

Theorem

Suppose $F = \mathbb{Q}$, $\mathbb{Q}(\pi)$ CM field. There is an embedding $\iota : D \to \mathrm{End}^0(B)$ if and only if:

1. $B \sim E^4$, where E is an elliptic curve.

$F = \mathbb{Q}$

Theorem

Suppose $F = \mathbb{Q}$, $\mathbb{Q}(\pi)$ CM field. There is an embedding $\iota : D \to \mathrm{End}^0(B)$ if and only if:

1. $B \sim E^4$, where E is an elliptic curve.
2. $B \sim S^2$, S simple surface, with either

$F = \mathbb{Q}$

Theorem

Suppose $F = \mathbb{Q}$, $\mathbb{Q}(\pi)$ CM field. There is an embedding $\iota : D \to \mathrm{End}^0(B)$ if and only if:

1. $B \sim E^4$, where E is an elliptic curve.
2. $B \sim S^2$, S simple surface, with either
   a. $\mathbb{Q}(\pi) = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, $\mathrm{End}^0(S)$ is a quaternion algebra, $D \otimes_{\mathbb{Q}} \mathbb{Q}(\pi) \simeq \mathrm{End}^0(S)$, and $[\mathbb{F}_q : \mathbb{F}_p]$ is even.

$F = \mathbb{Q}$

Theorem

Suppose $F = \mathbb{Q}$, $\mathbb{Q}(\pi)$ CM field. There is an embedding $\iota : D \to \mathrm{End}^0(B)$ if and only if:

1. $B \sim E^4$, where E is an elliptic curve.
2. $B \sim S^2$, S simple surface, with either
   a. $\mathbb{Q}(\pi) = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, $\mathrm{End}^0(S)$ is a quaternion algebra, $D \otimes_{\mathbb{Q}} \mathbb{Q}(\pi) \simeq \mathrm{End}^0(S)$, and $[\mathbb{F}_q : \mathbb{F}_p]$ is even.
   b. $\mathrm{End}^0(S) = \mathbb{Q}(\pi)$ a quartic CM field that splits D. If in addition D ramifies at p, then S is ordinary or supersingular.

$F = \mathbb{Q}$

Theorem

Suppose $F = \mathbb{Q}$, $\mathbb{Q}(\pi)$ CM field. There is an embedding $\iota : D \to \mathrm{End}^0(B)$ if and only if:

1. $B \sim E^4$, where E is an elliptic curve.
2. $B \sim S^2$, S simple surface, with either
   a. $\mathbb{Q}(\pi) = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, $\mathrm{End}^0(S)$ is a quaternion algebra, $D \otimes_{\mathbb{Q}} \mathbb{Q}(\pi) \simeq \mathrm{End}^0(S)$, and $[\mathbb{F}_q : \mathbb{F}_p]$ is even.
   b. $\mathrm{End}^0(S) = \mathbb{Q}(\pi)$ a quartic CM field that splits D. If in addition D ramifies at p, then S is ordinary or supersingular.
3. B is a supersingular simple fourfold, $\mathbb{Q}(\pi)$ is a quartic CM field, $\mathrm{End}^0(B) \simeq D \otimes_{\mathbb{Q}} \mathbb{Q}(\pi)$, and D ramifies at p.

Epilogue: Problem A' for abelian fourfolds

Epilogue: Problem A' for abelian surfaces

Theorem (F.)

Let k number field, A/k simple abelian fourfold such that $\mathrm{End}^0(A)$ is a quaternion algebra.
Then, A splits modulo all but finitely many primes.

Problem A'

- Can we say anything more about the set of simple reductions?
- Is it ever nonempty?

Epilogue: Problem A' for abelian surfaces

Theorem

Let k number field, A/k a simple abelian fourfold such that $\mathrm{End}^0(A)$ is a quaternion algebra **over** $\mathbb{Q}$.

Epilogue: Problem A' for abelian surfaces

Theorem

Let k number field, A/k a simple abelian fourfold such that $\mathrm{End}^0(A)$ is a quaternion algebra **over** $\mathbb{Q}$.
For every $\mathfrak{p} \in \Sigma_A$, $A_{\mathfrak{p}}$ is split or supersingular.

Epilogue: Problem A' for abelian surfaces

Theorem

Let k number field, A/k a simple abelian fourfold such that $\mathrm{End}^0(A)$ is a quaternion algebra **over** $\mathbb{Q}$.
For every $\mathfrak{p} \in \Sigma_A$, $A_{\mathfrak{p}}$ is split or supersingular.
In particular, $A_{\mathfrak{p}}$ is always **geometrically split**.

Epilogue: Problem A' for abelian surfaces

Theorem

Let k number field, A/k a simple abelian fourfold such that $\mathrm{End}^0(A)$ is a quaternion algebra **over** $\mathbb{Q}$.
For every $\mathfrak{p} \in \Sigma_A$, $A_{\mathfrak{p}}$ is split or supersingular.
In particular, $A_{\mathfrak{p}}$ is always **geometrically split**.

But:

- There exists an abelian fourfold A over $\mathbb{Q}(\sqrt{-1}, \sqrt{-3}, \sqrt{13})$,

Epilogue: Problem A' for abelian surfaces

Theorem

Let k number field, A/k a simple abelian fourfold such that $\mathrm{End}^0(A)$ is a quaternion algebra **over** $\mathbb{Q}$.
For every $\mathfrak{p} \in \Sigma_A$, $A_{\mathfrak{p}}$ is split or supersingular.
In particular, $A_{\mathfrak{p}}$ is always **geometrically split**.

But:

- There exists an abelian fourfold A over $\mathbb{Q}(\sqrt{-1}, \sqrt{-3}, \sqrt{13})$,
- with $\mathrm{End}^0(A)$ a quaternion algebra over $\mathbb{Q}(\sqrt{17})$,

Epilogue: Problem A' for abelian surfaces

Theorem

Let k number field, A/k a simple abelian fourfold such that $\mathrm{End}^0(A)$ is a quaternion algebra **over** $\mathbb{Q}$.
For every $\mathfrak{p} \in \Sigma_A$, $A_{\mathfrak{p}}$ is split or supersingular.
In particular, $A_{\mathfrak{p}}$ is always **geometrically split**.

But:

- There exists an abelian fourfold A over $\mathbb{Q}(\sqrt{-1}, \sqrt{-3}, \sqrt{13})$,
- with $\mathrm{End}^0(A)$ a quaternion algebra over $\mathbb{Q}(\sqrt{17})$,
- which is **geometrically simple** modulo $\mathfrak{p} \mid 17$.

Quaternionic multiplication and abelian fourfolds

Enric Florit

Universitat de Barcelona

The SQIparty 2025