Threshold signatures from different group actions

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2025.04.30 - SQIparty - Lleida SPAIN

Universität Zürich[UZH]
IBM

- Introduction of different *group actions*

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- N-out-of-N case

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- Active security

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- Few words on open problems and DKG

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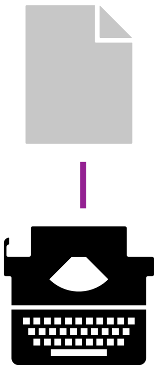

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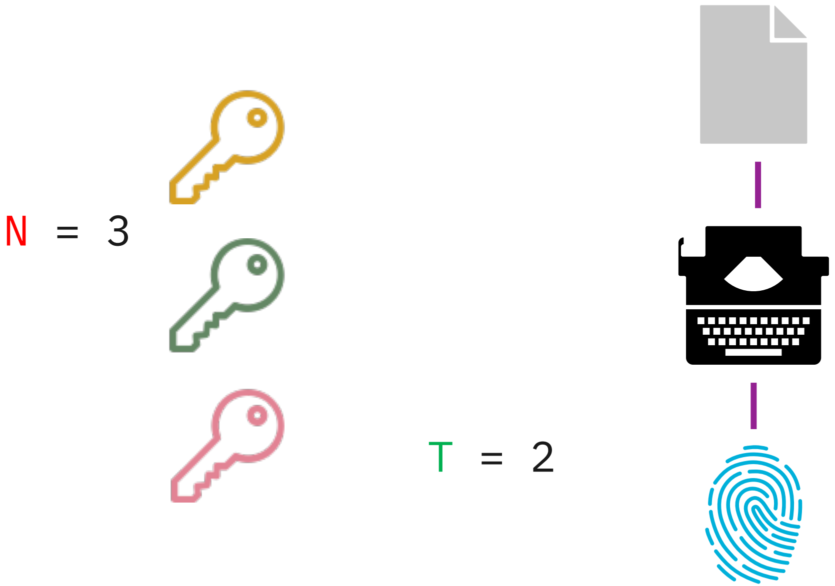

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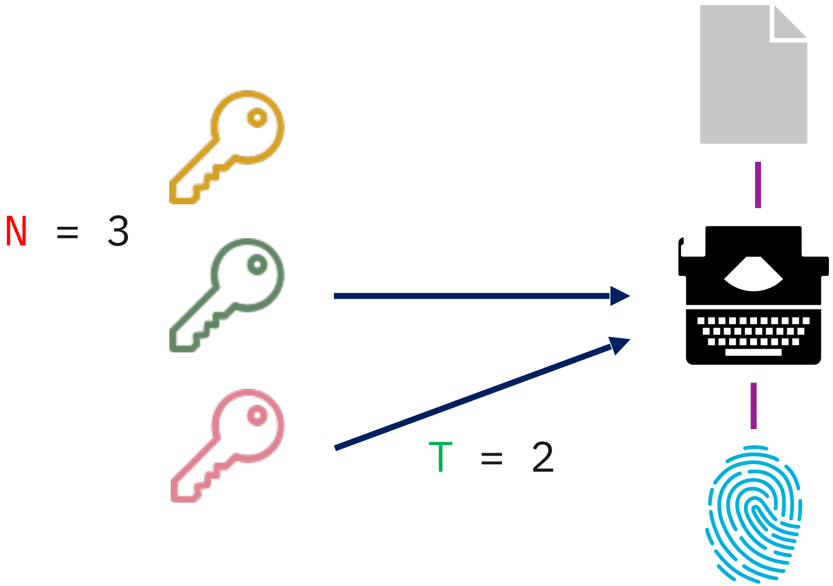

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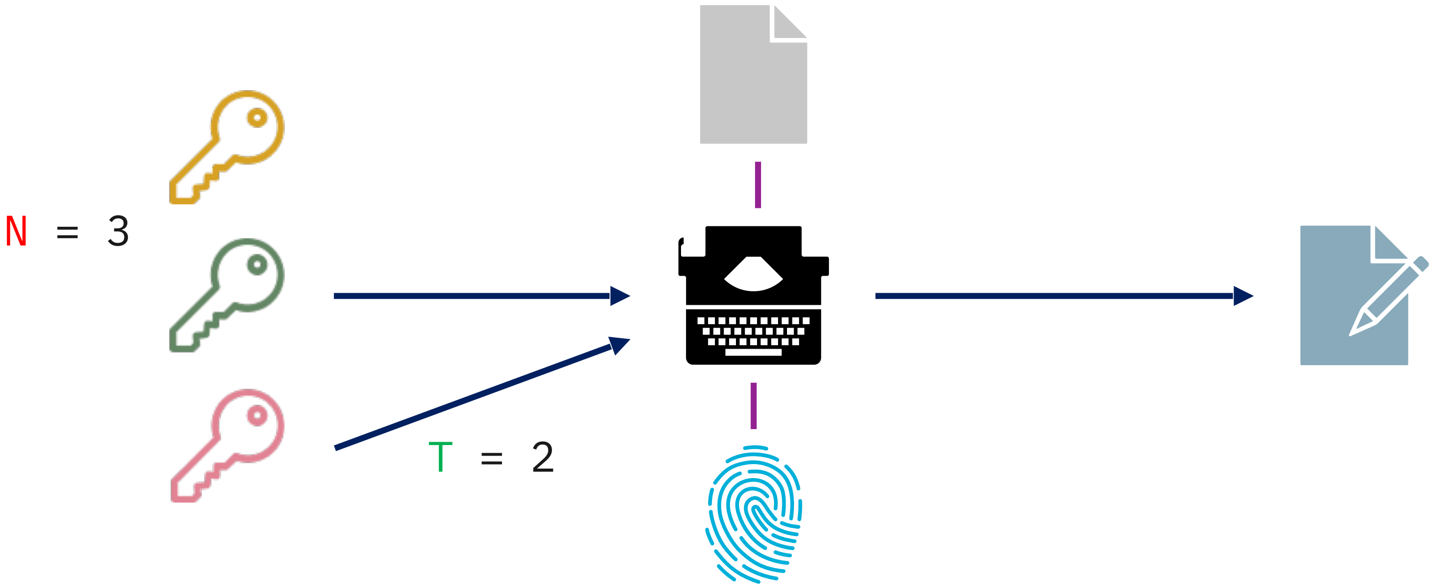

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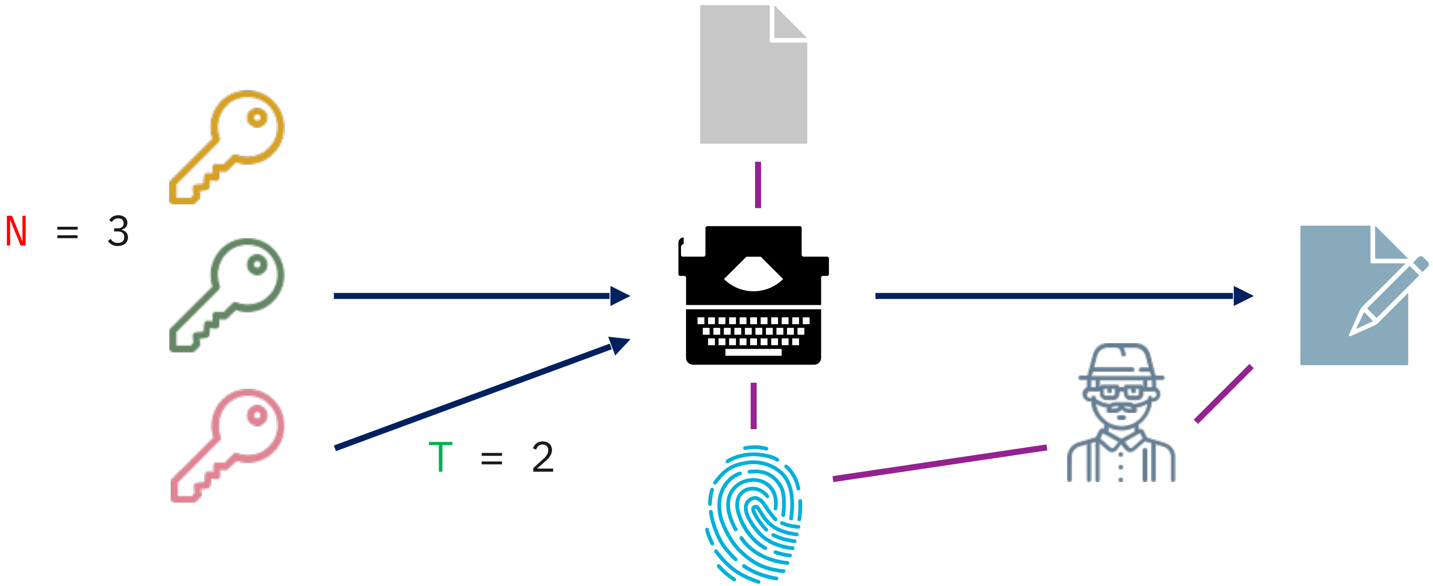

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  - Parallelisation: given $x, y = g \star x, z = h \star x$ and w it is hard to say if $w = (gh) \star x$

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  - PEGASIS (3) > **Abelian**

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(3) Dartois P, Eriksen JK, Fouotsa TB, Le Merdy AH, Invernizzi R, Robert D, Rueger R, Vercauteren F, Wesolowski B. PEGASIS: Practical Effective Class Group Action using 4-Dimensional Isogenies.

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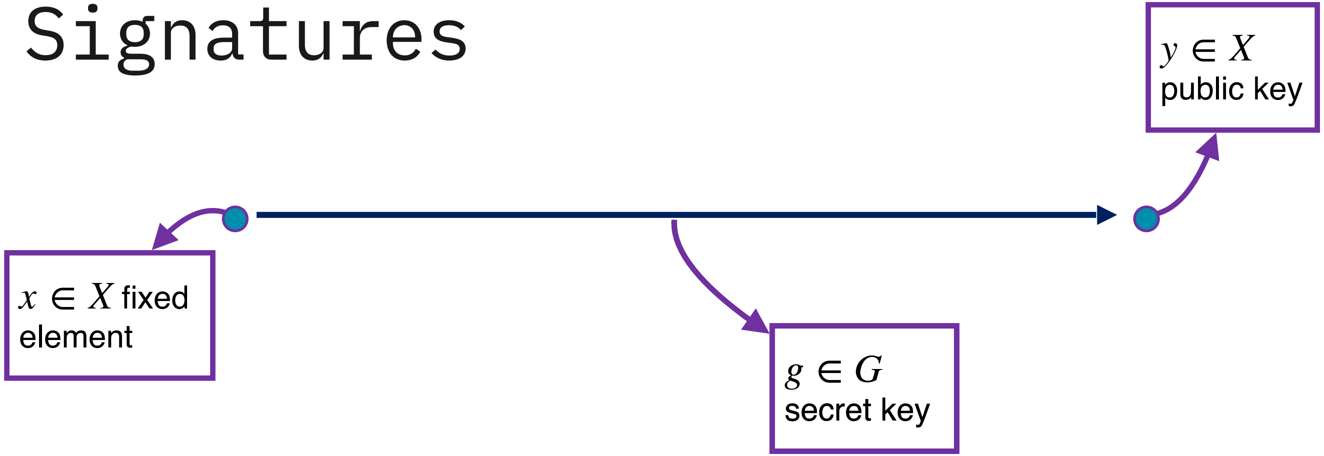

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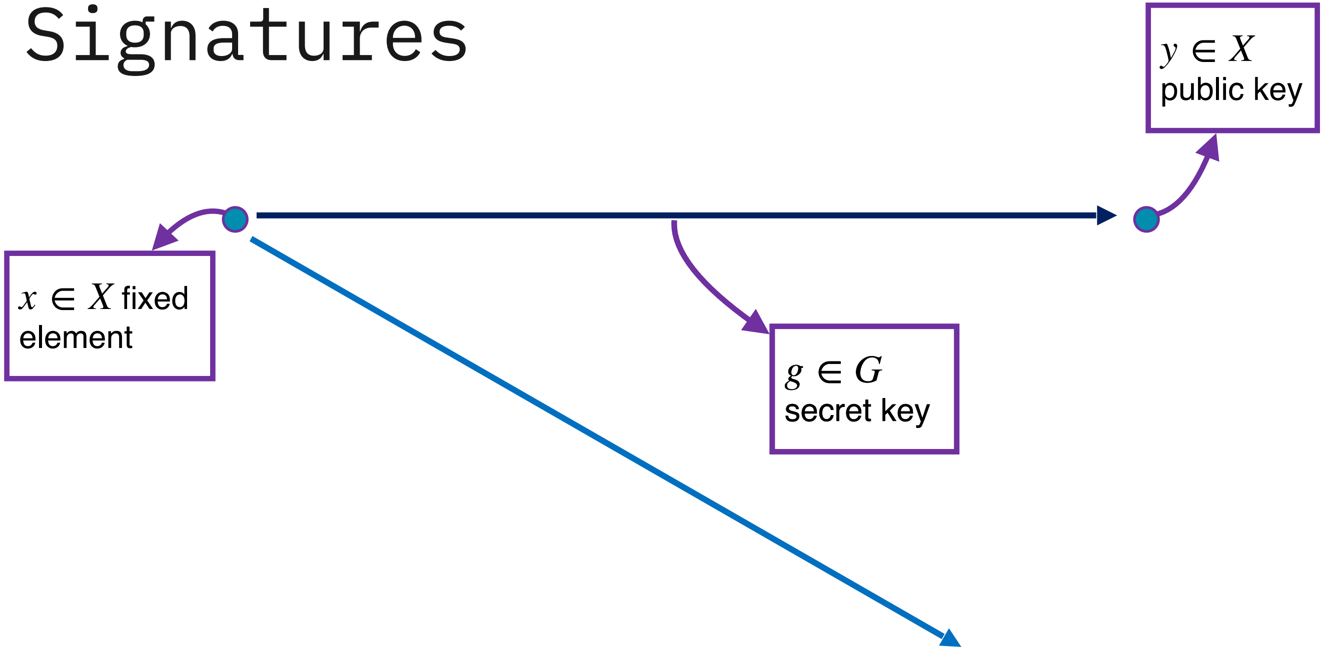

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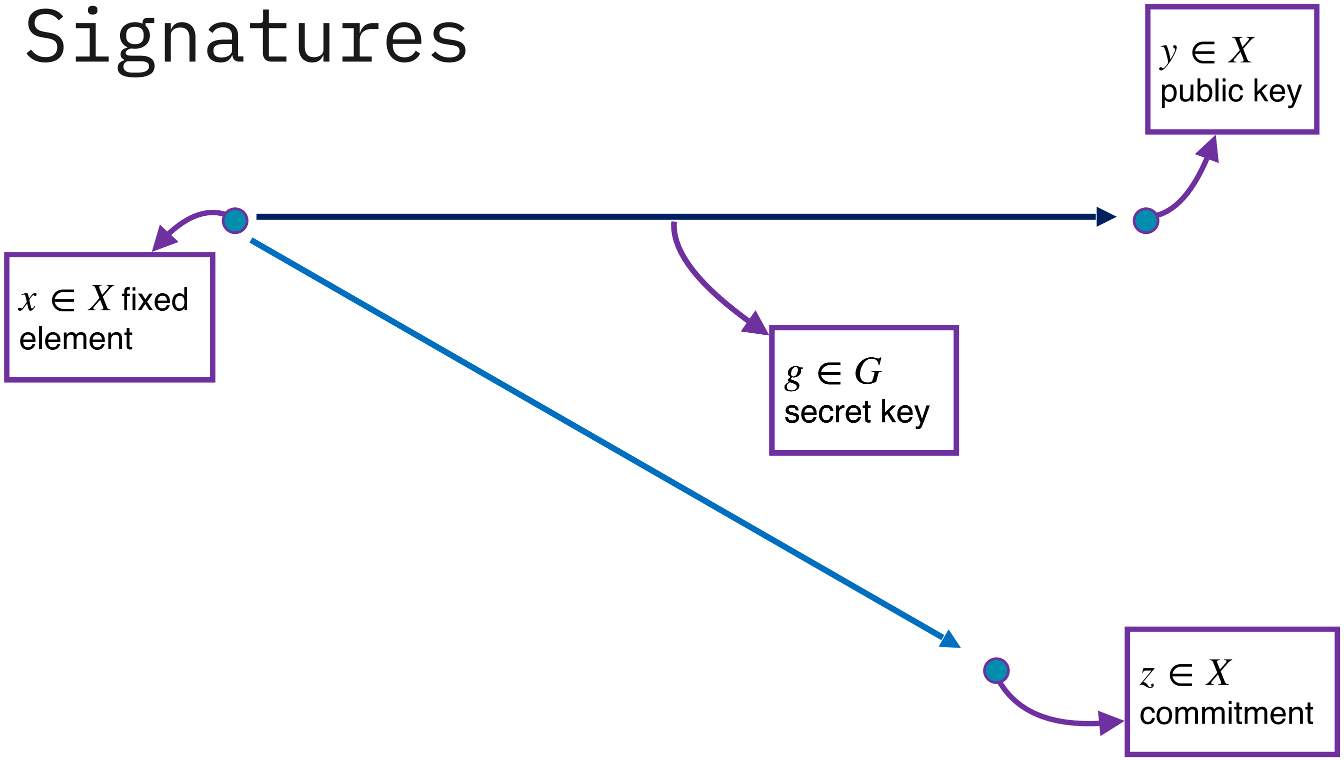

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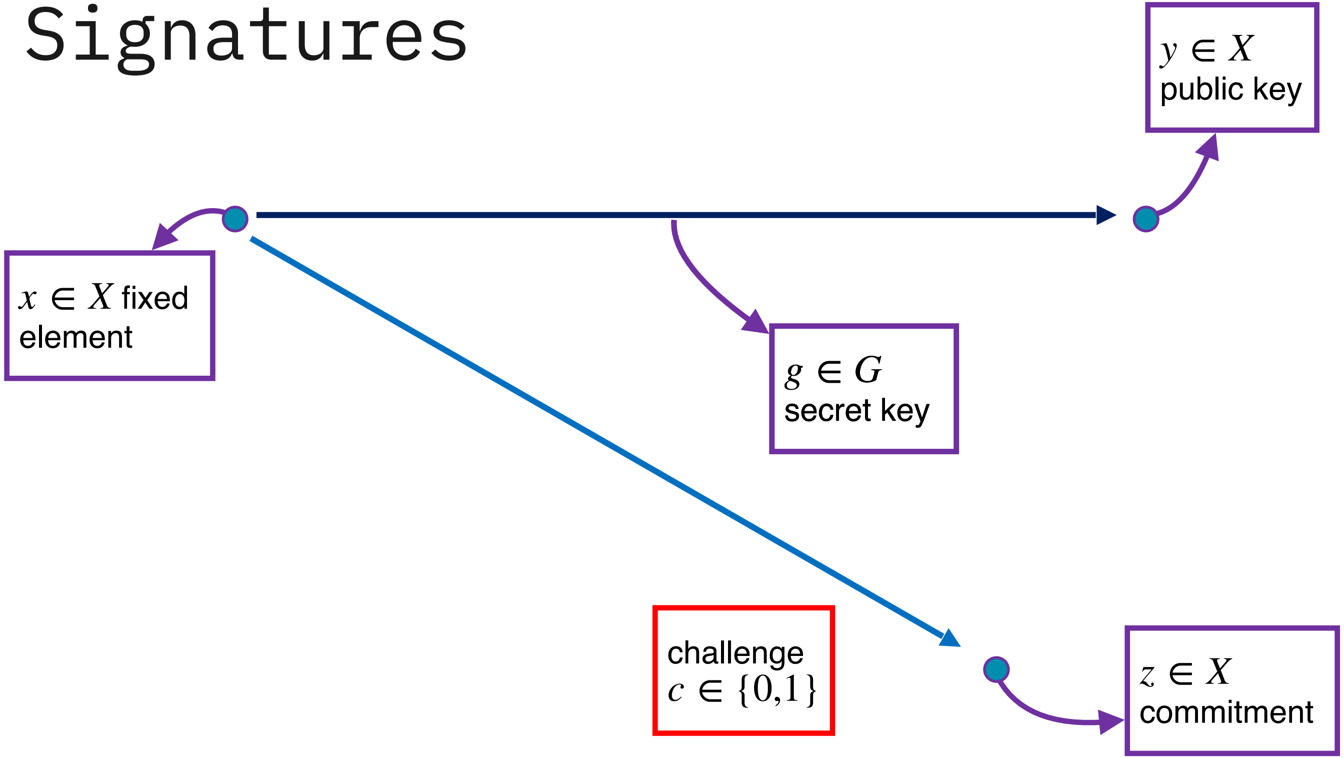

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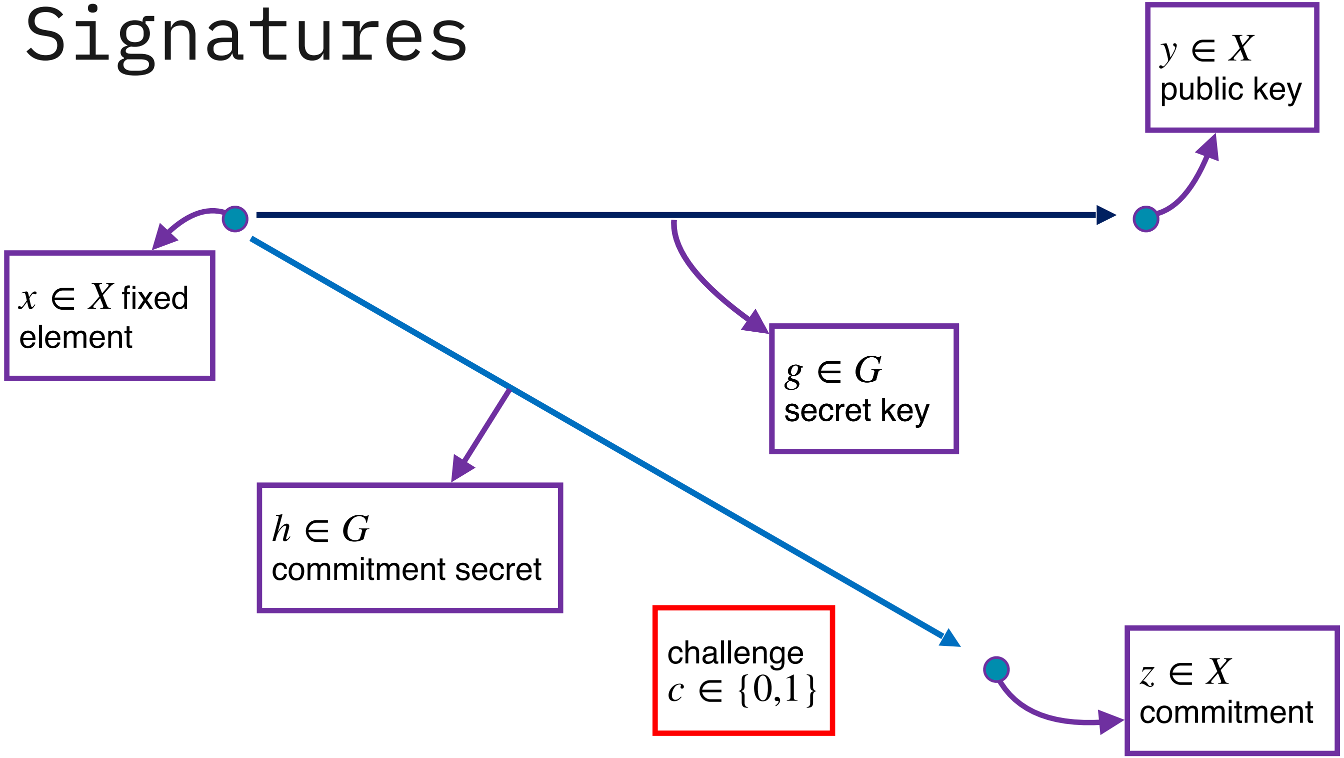

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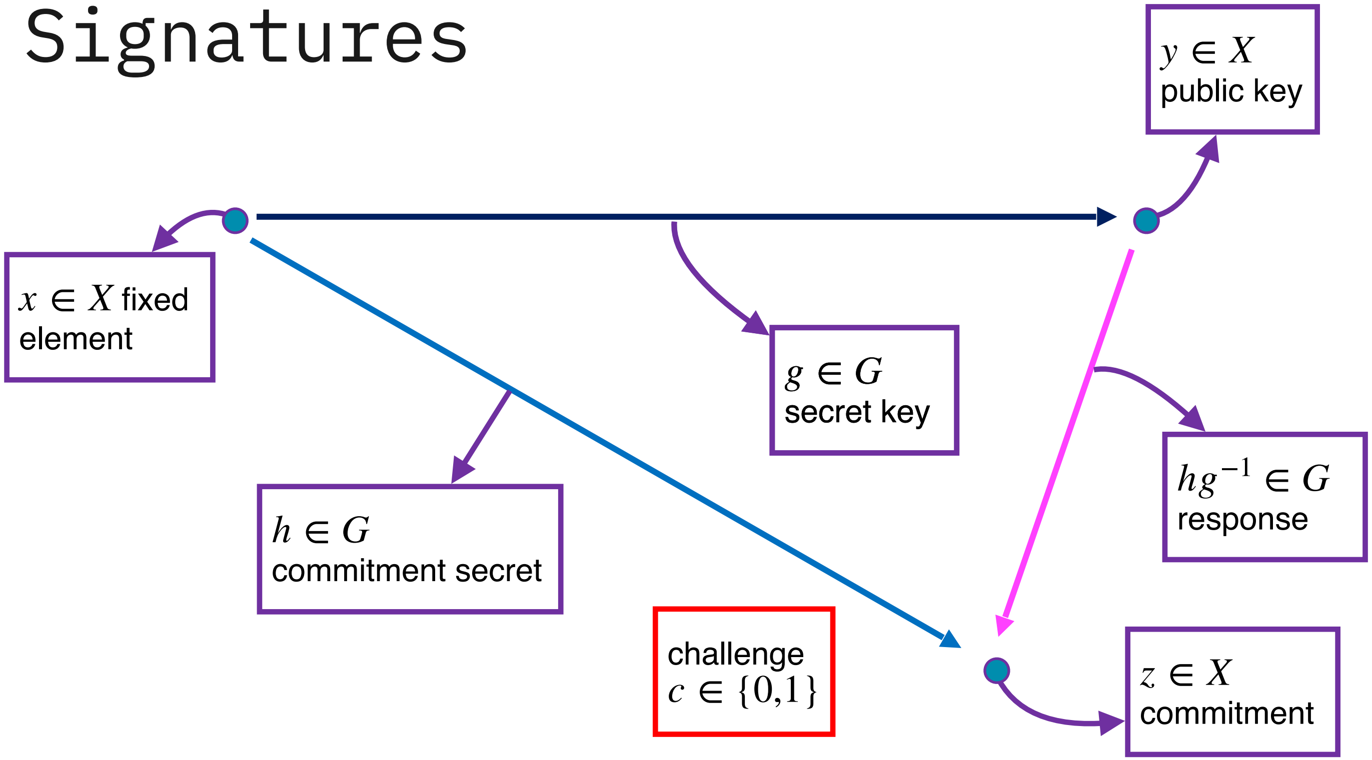

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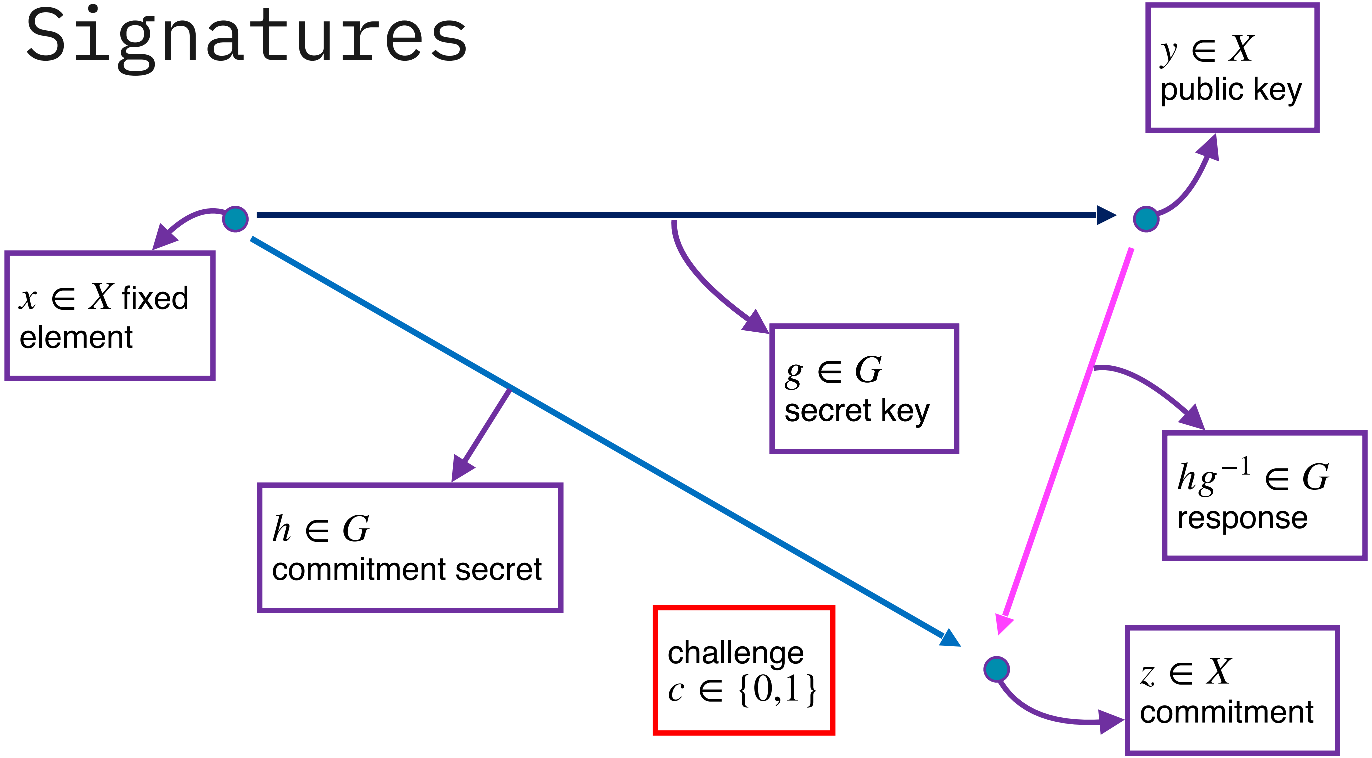

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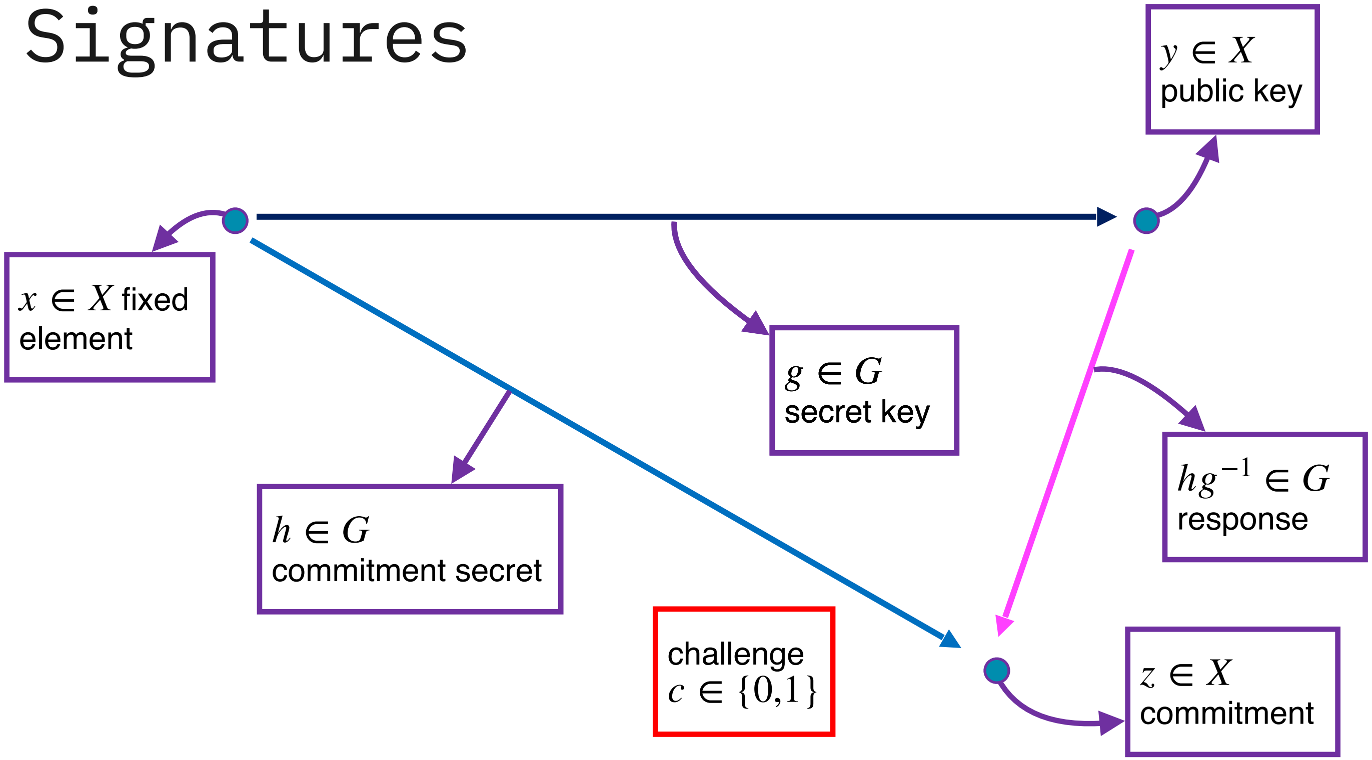

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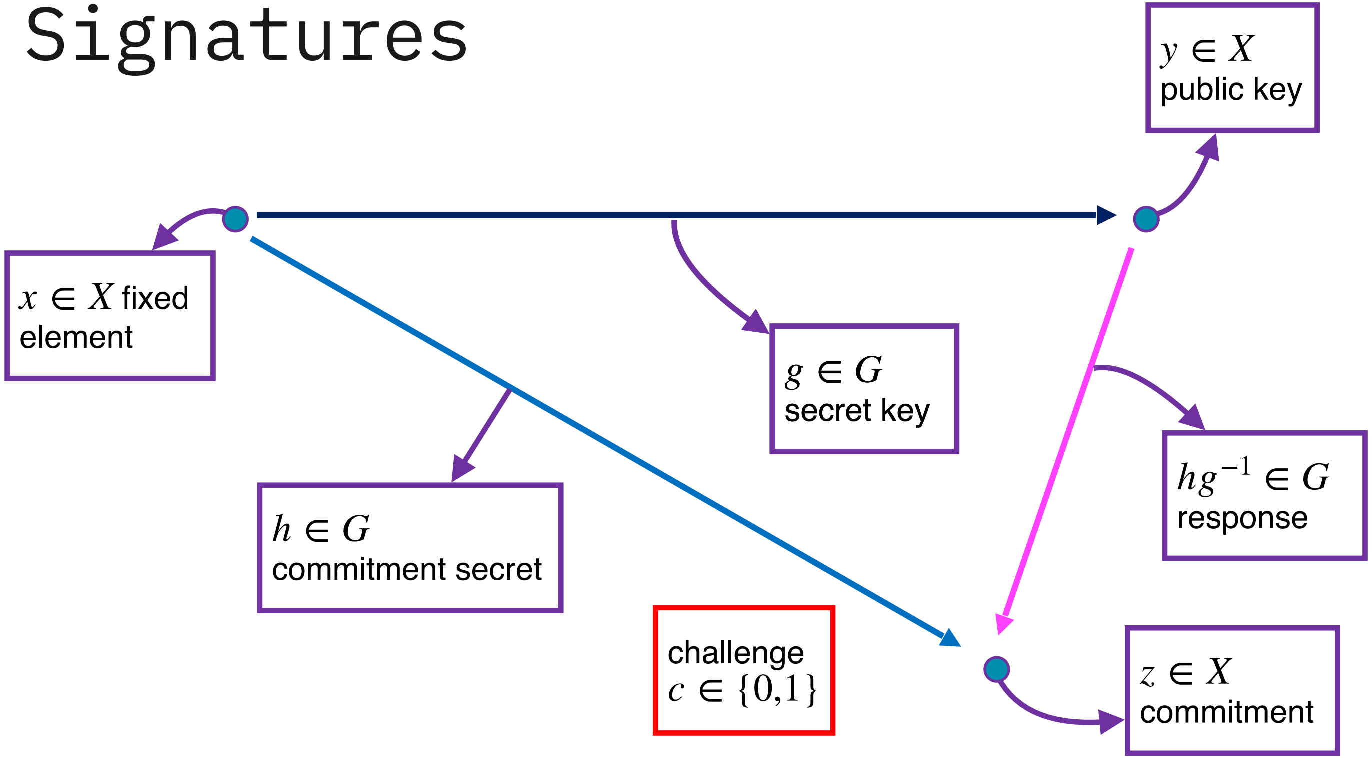

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-Boneh et.al. (2): you need to do that at least λ group actions.

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(2) Boneh D, Guan J, Zhandry M. A lower bound on the length of signatures based on group actions and generic isogenies.

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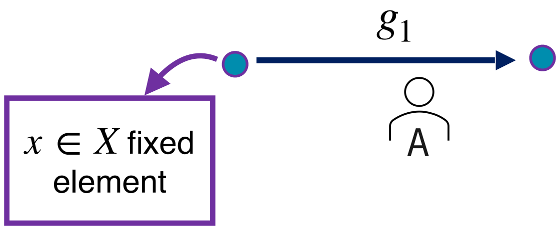

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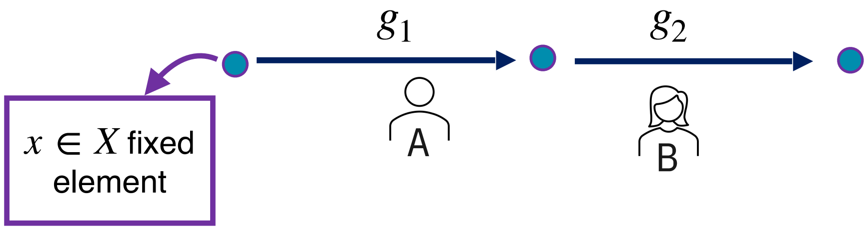

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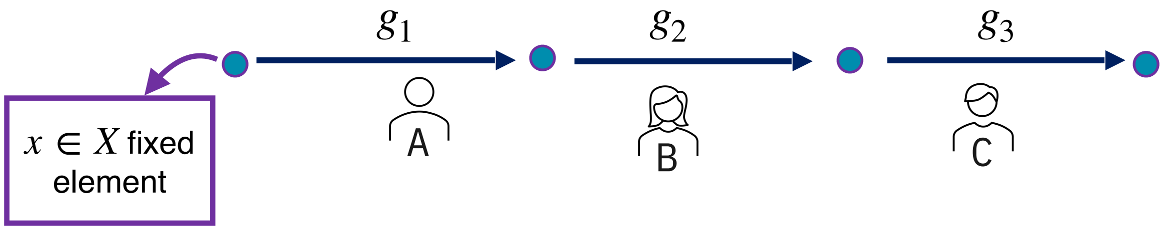

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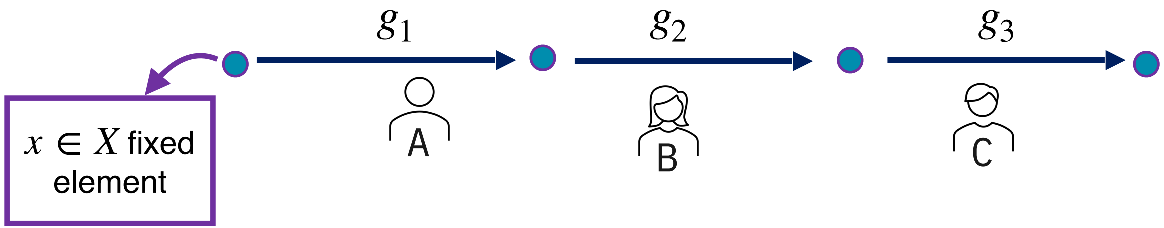
$g = g_N \cdots g_2 \cdot g_1$
shared secret key



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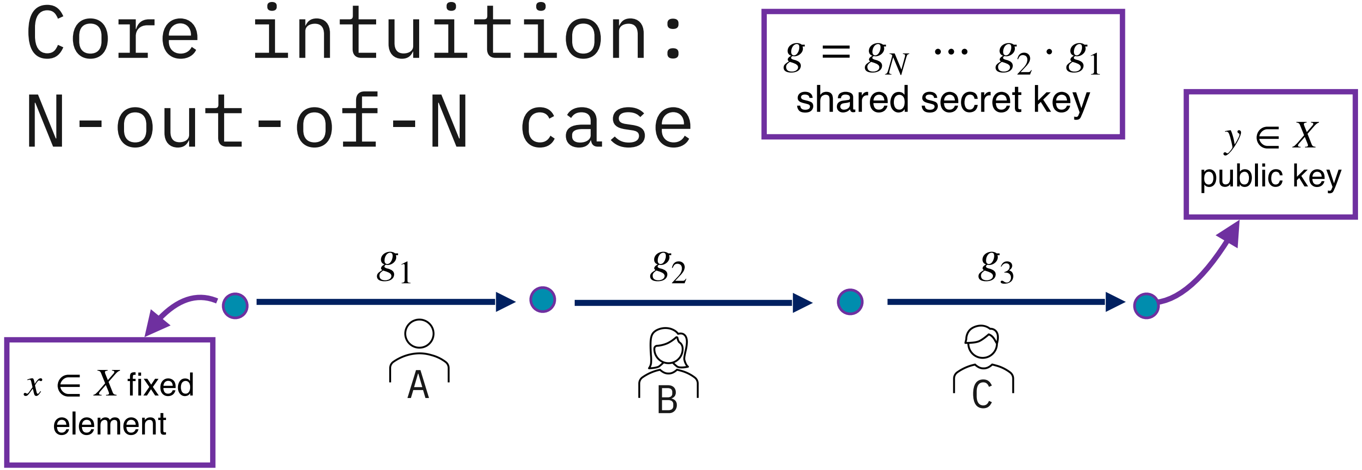

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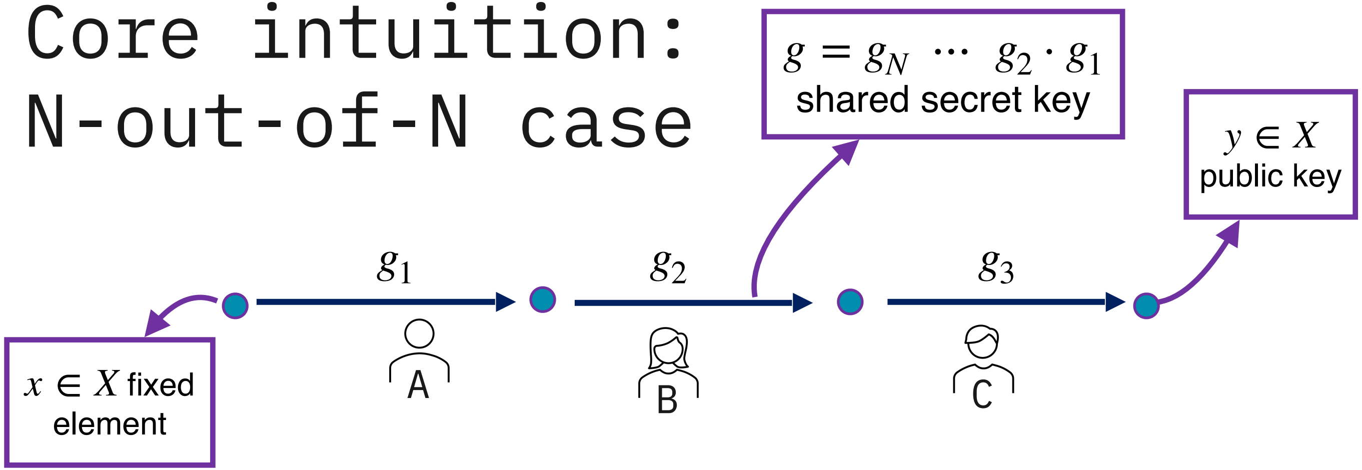

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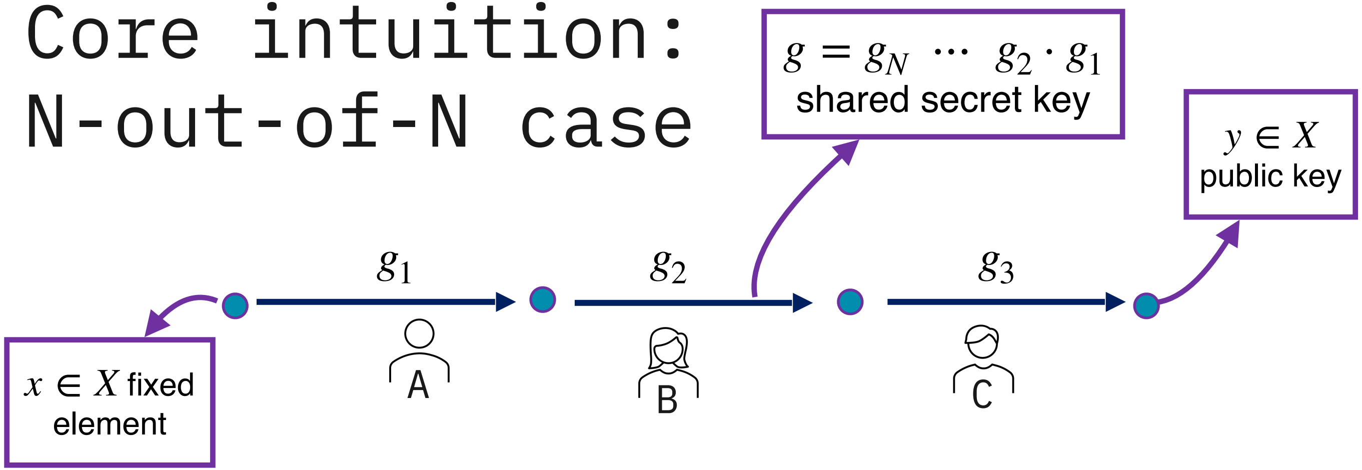

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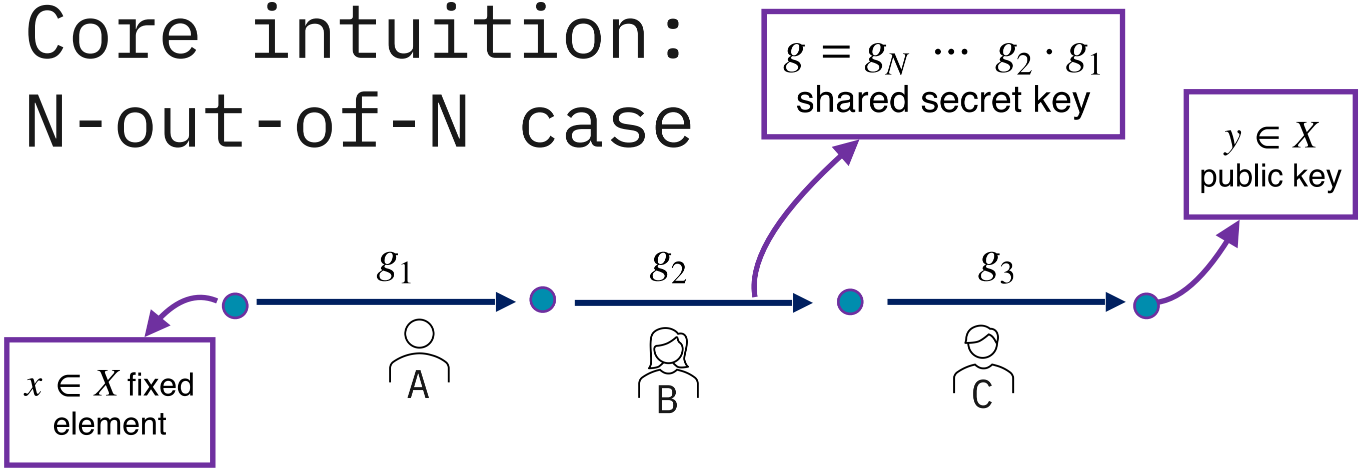

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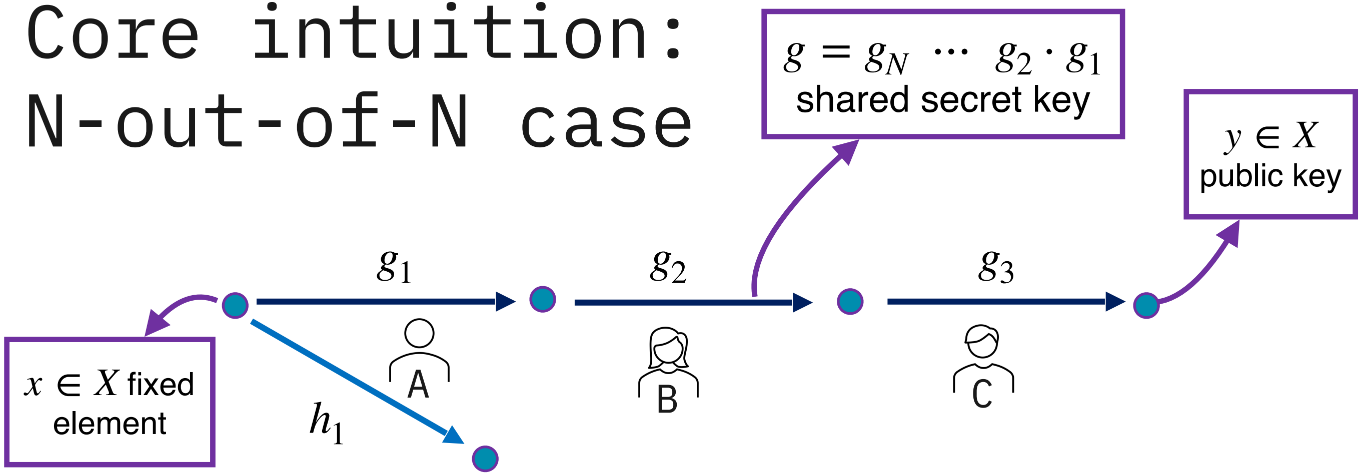

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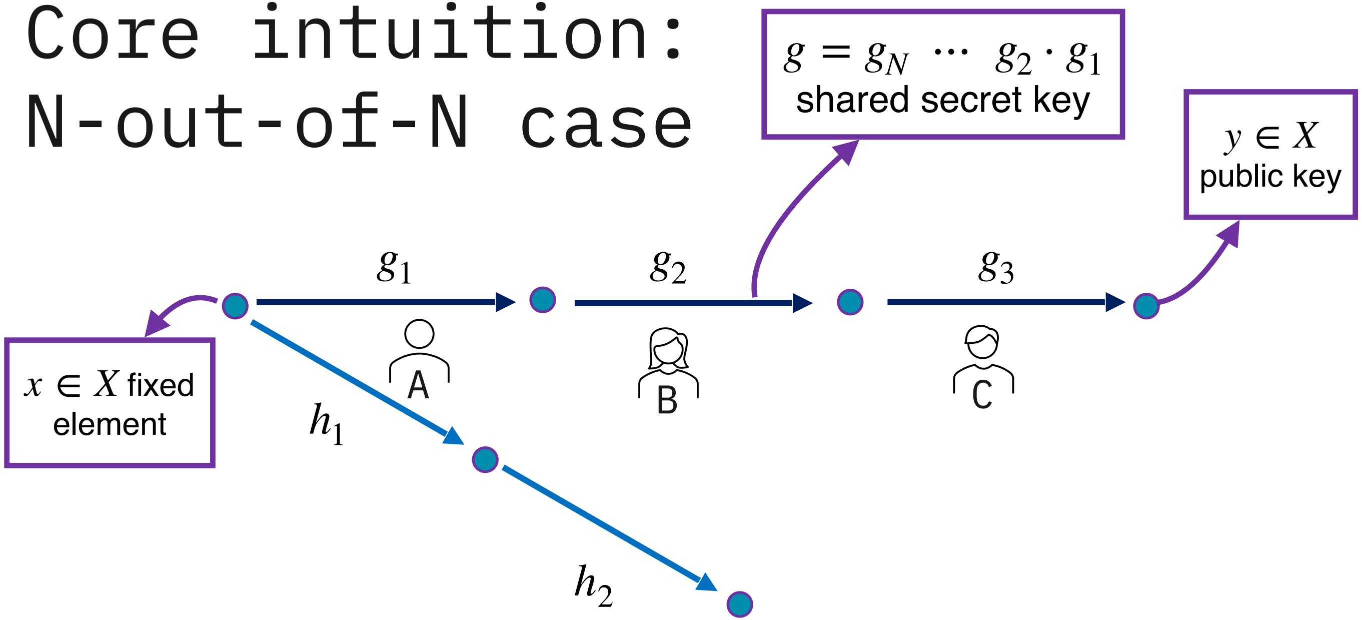

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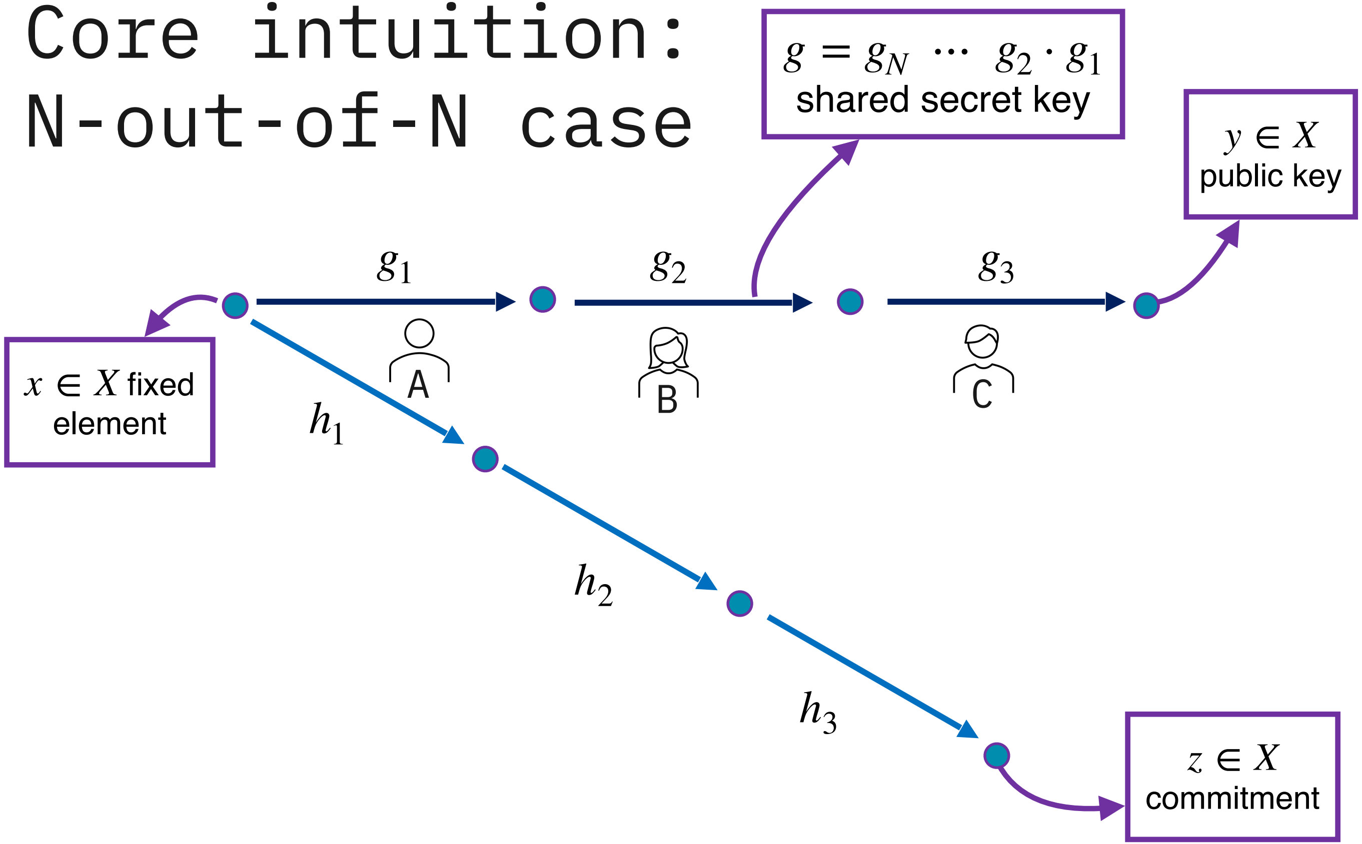

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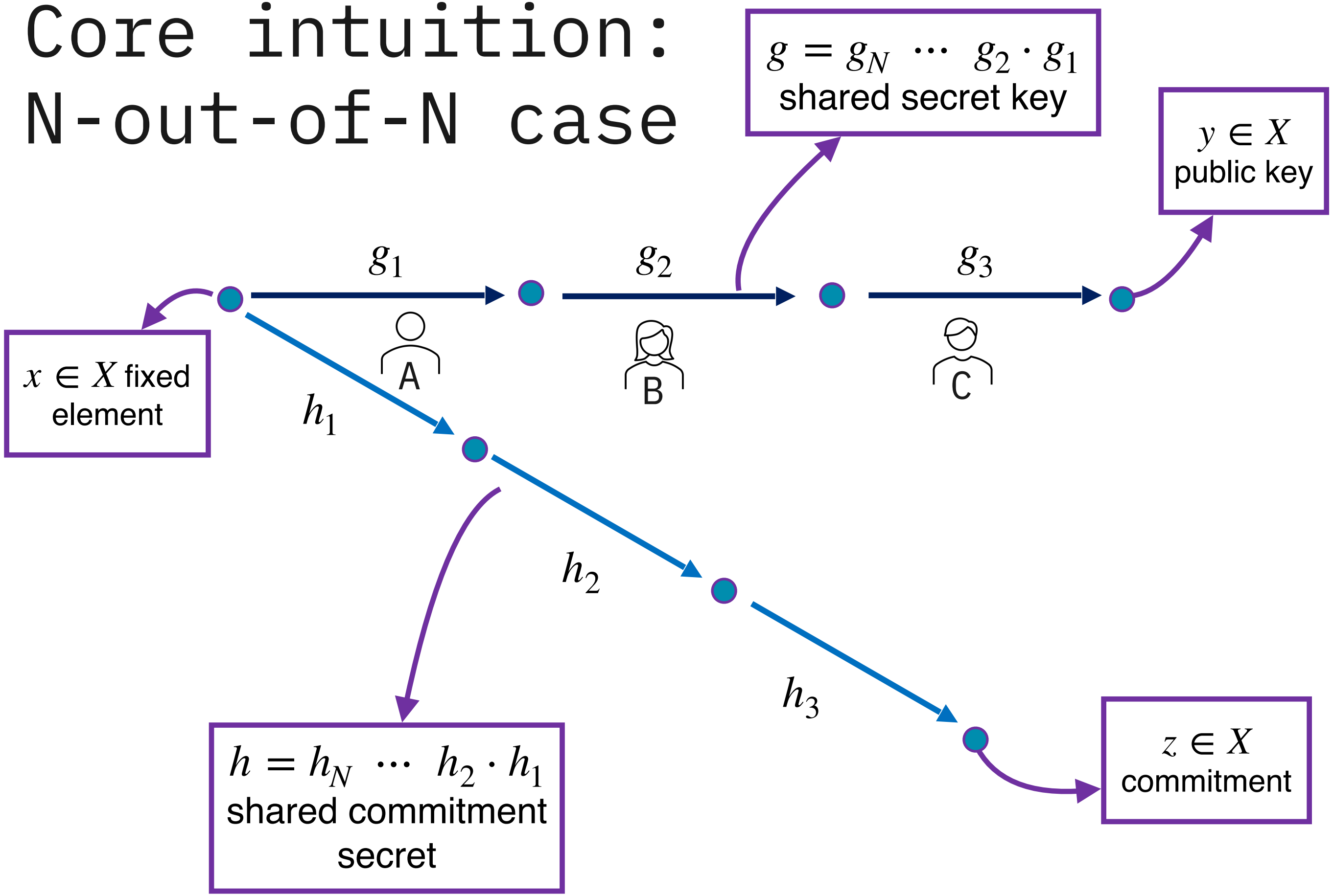

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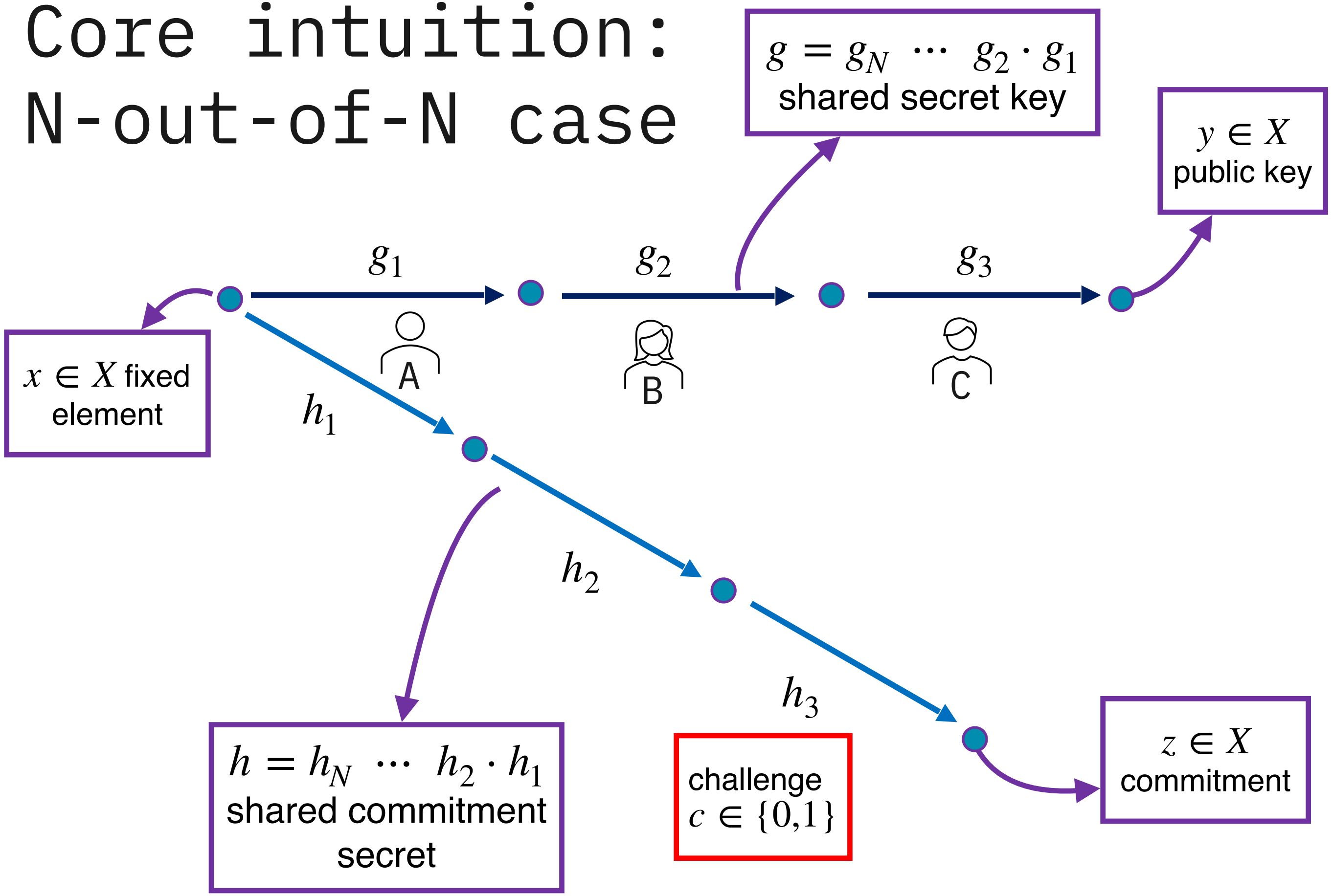

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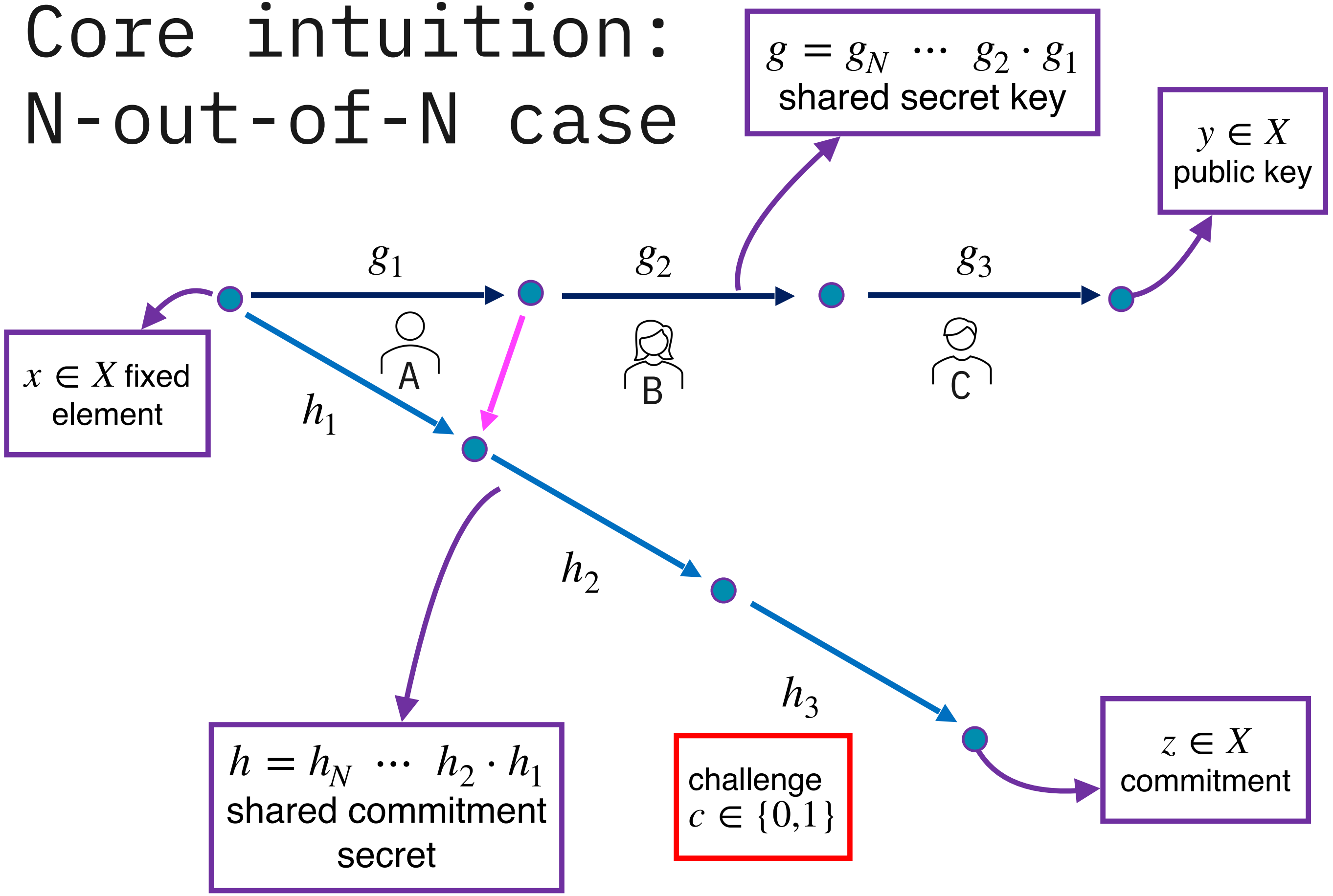

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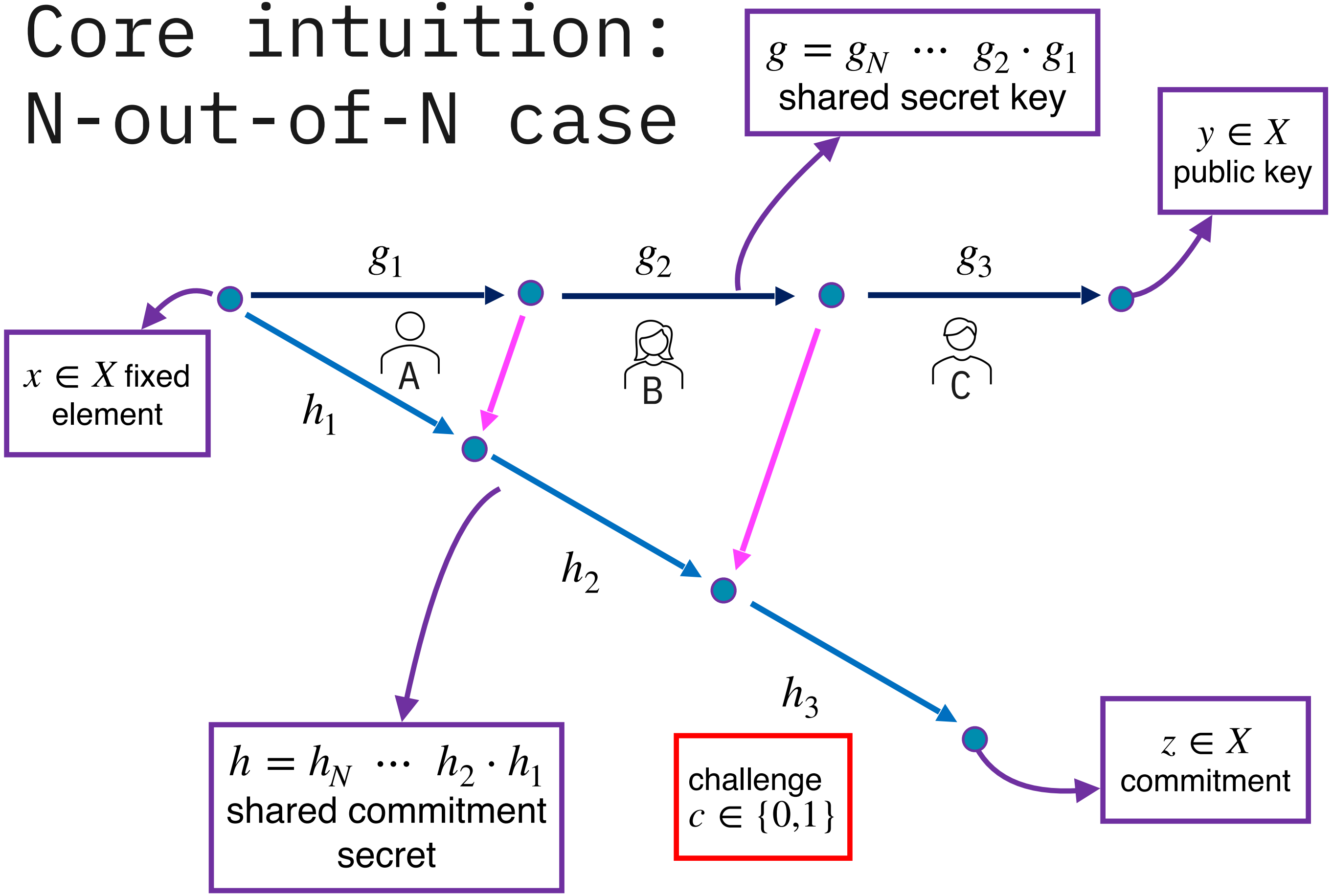

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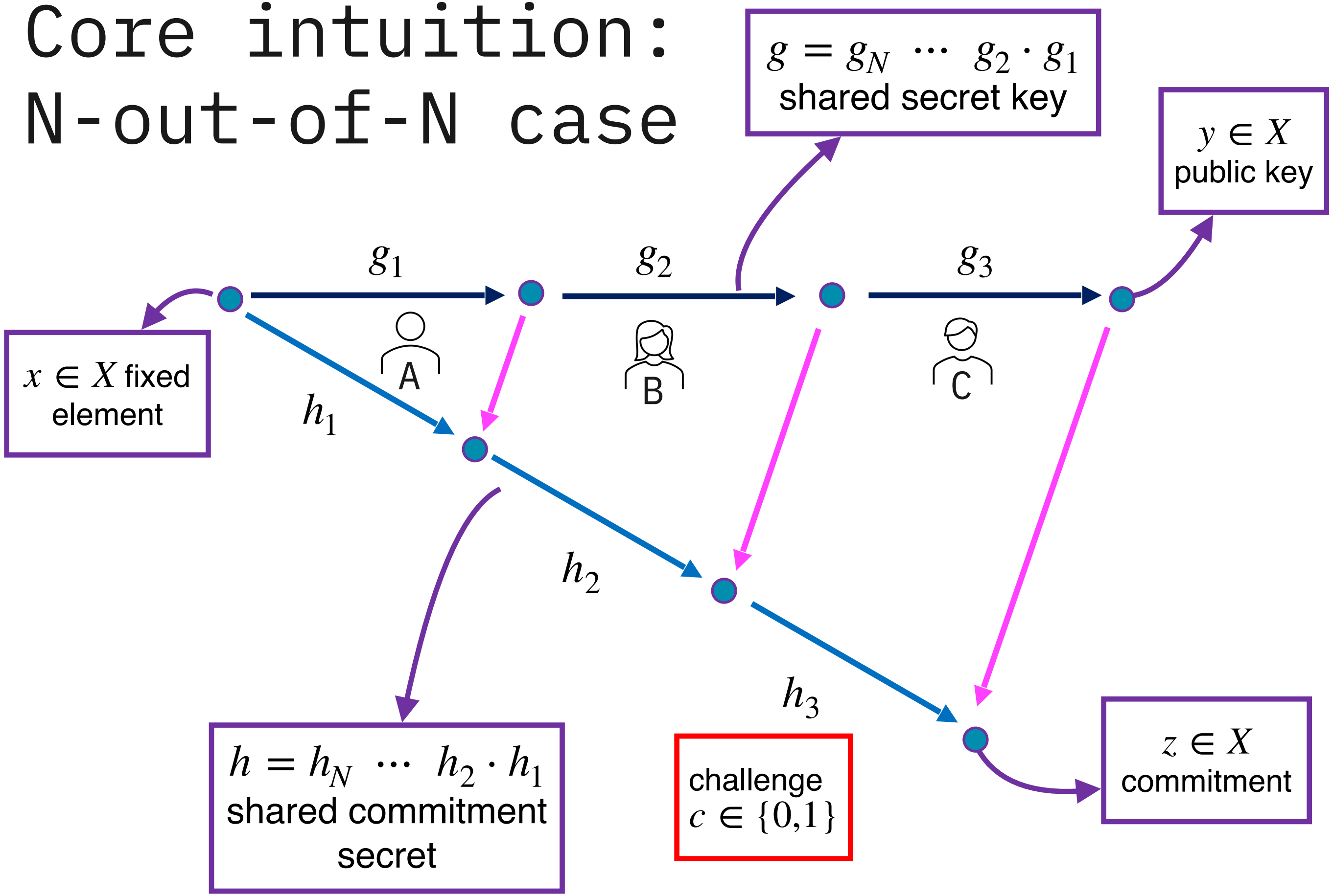

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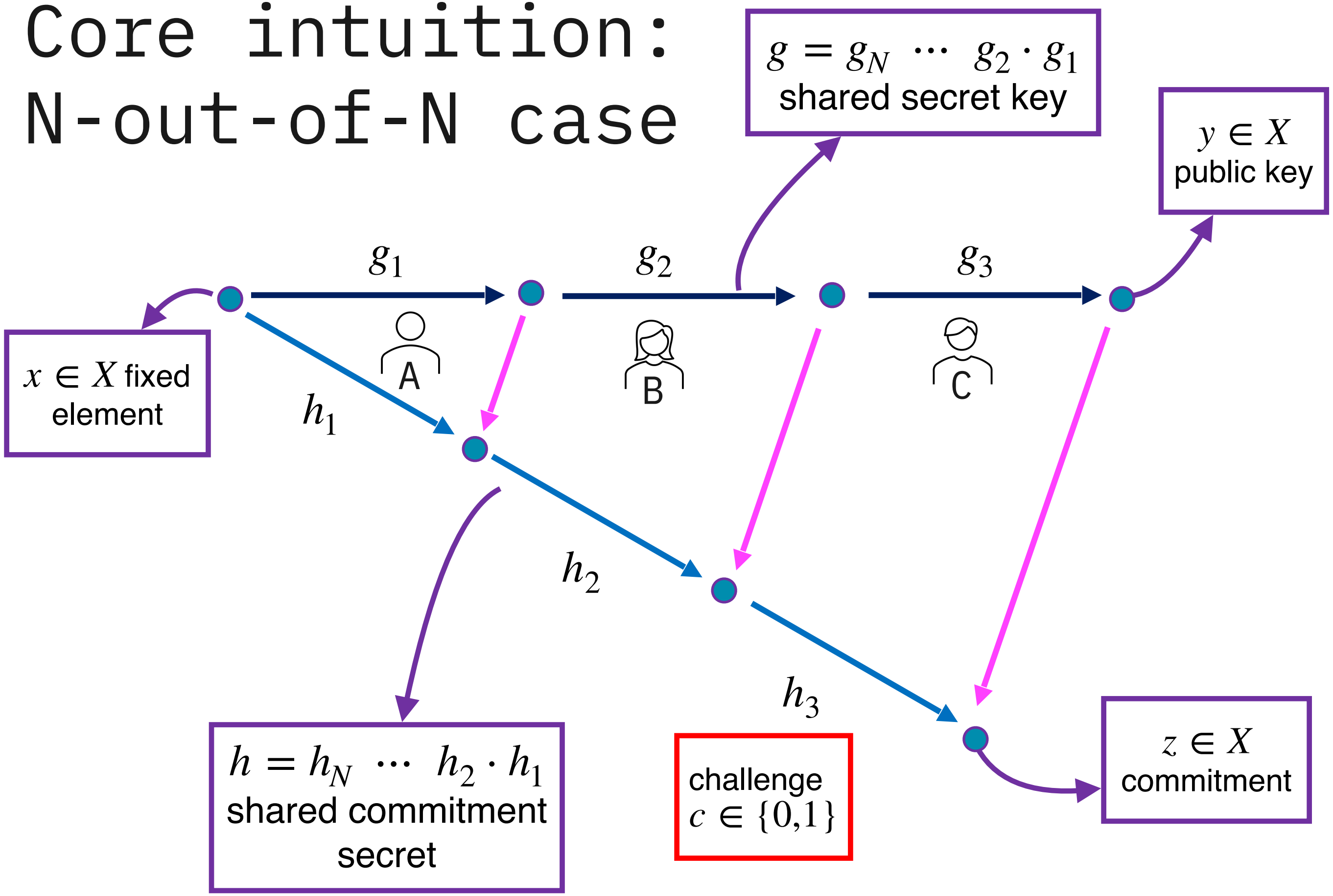

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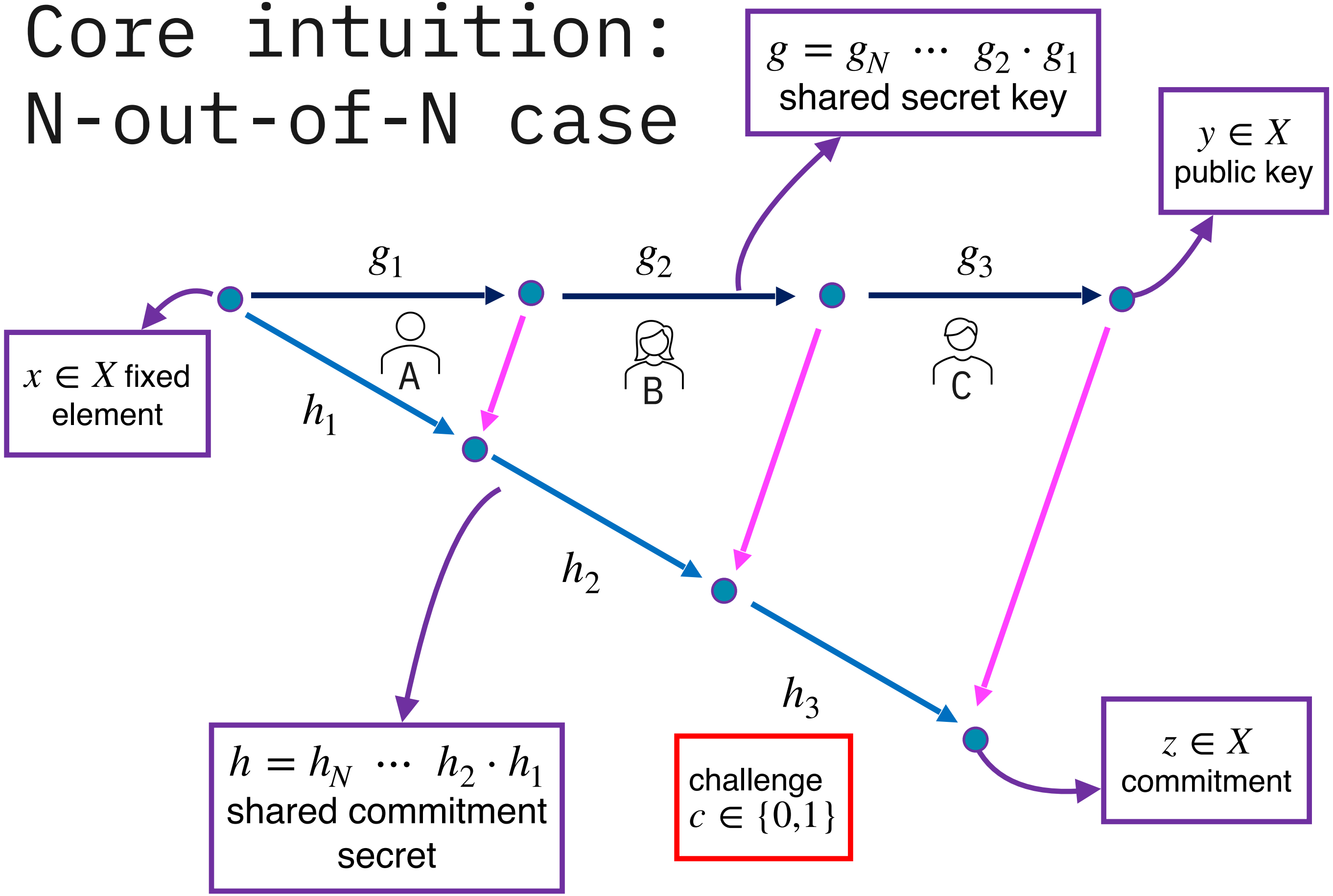

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- the hard part is the sharing of the secret, not the commitment

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  - **Pro**: Much more efficient;
  - **Con**: Requires to know all intermediate public keys.

Cozzo D, Giunta E. Round-robin is optimal: lower bounds for group action based protocols.

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# Rounds				
Complexity				
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	Passive, Non-Abelian			
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De Feo L, Meyer M. Threshold schemes from isogeny assumptions

How to make this for T-out-of-N ?

Cyclic Case

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Shamir Secret Sharing

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- **Problem 3**: ZKPs becomes much more complicated (PVP)

How to make this for T-out-of-N ?
<u>Non-Abelian Case</u>

Battagliola M, Borin G, Meneghetti A, Persichetti E. Cutting the grass: Threshold group action signature schemes.

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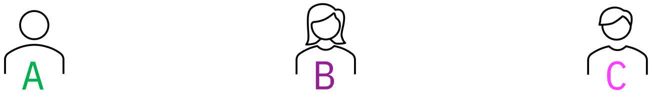

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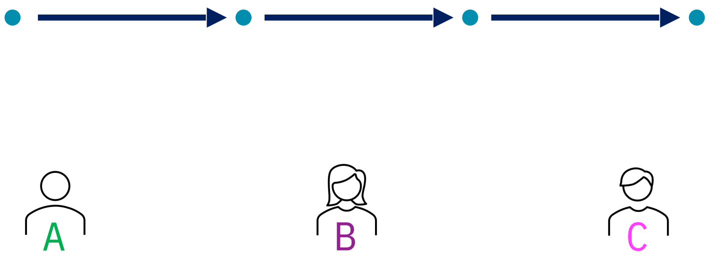

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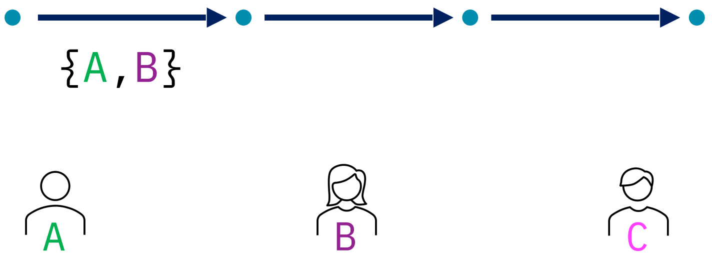

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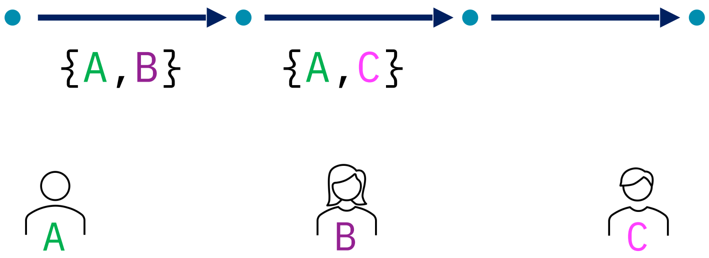

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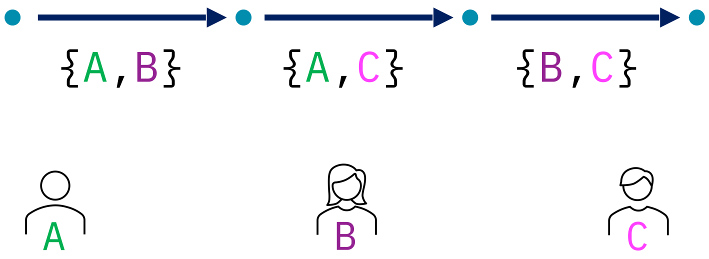

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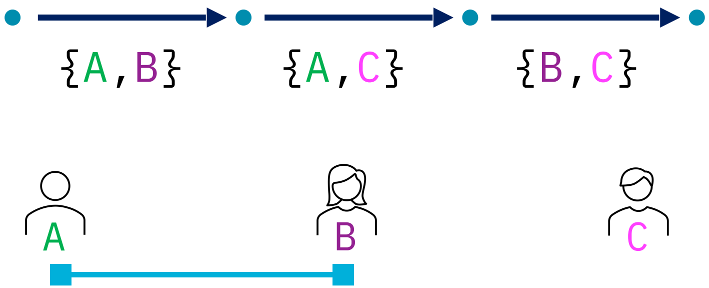

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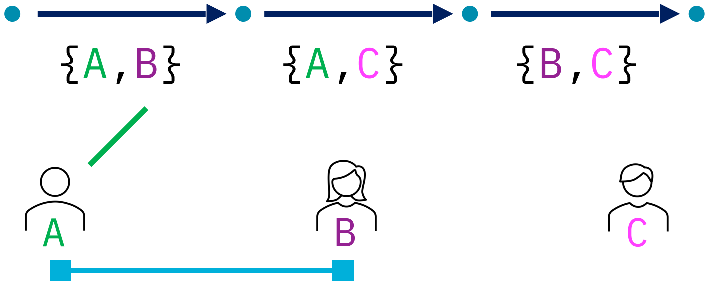

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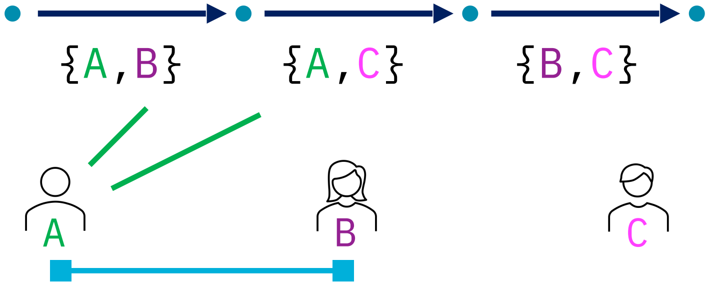

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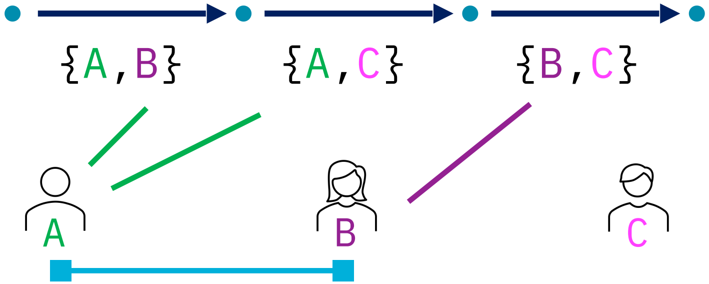

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‘Vandermonde’ Secret Sharing

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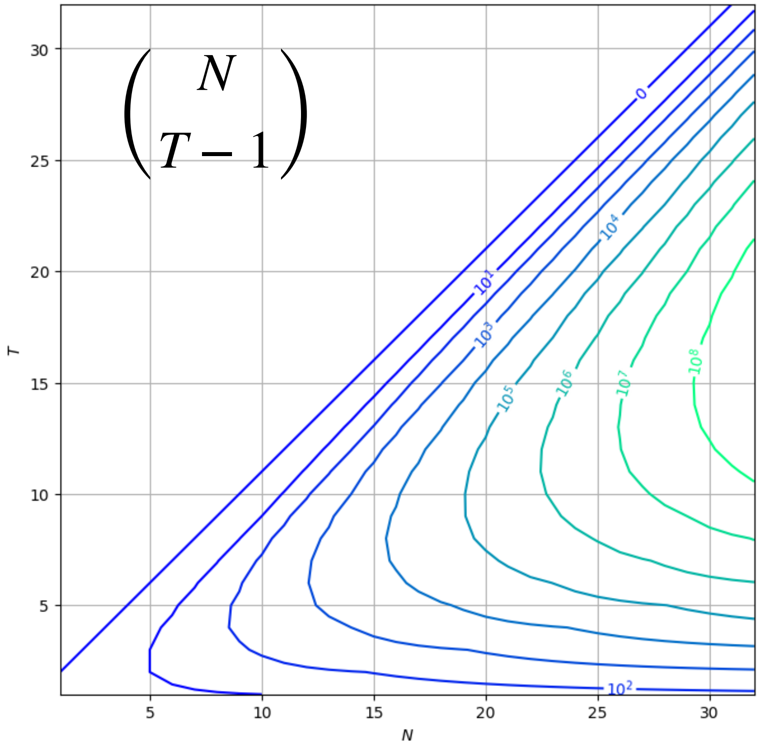

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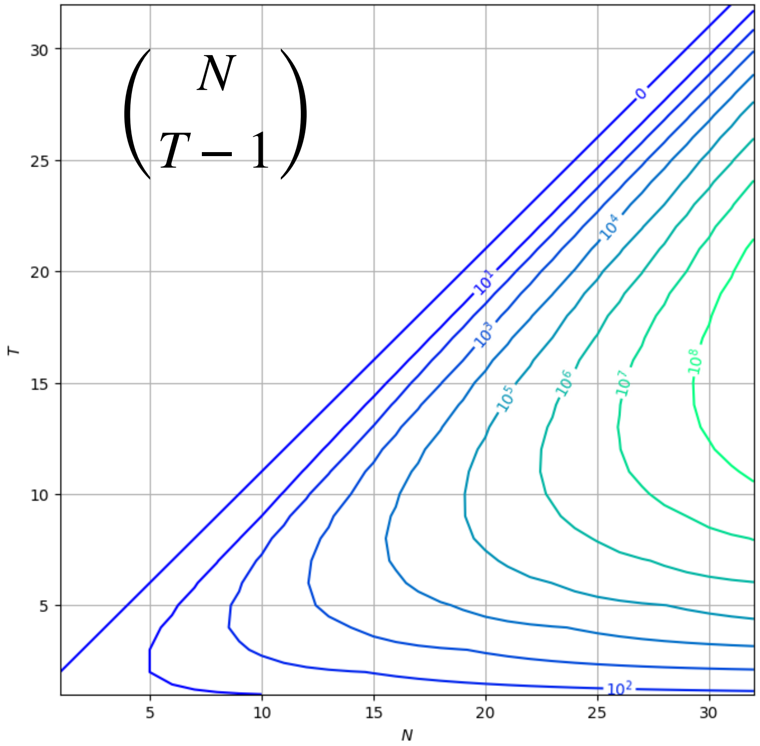

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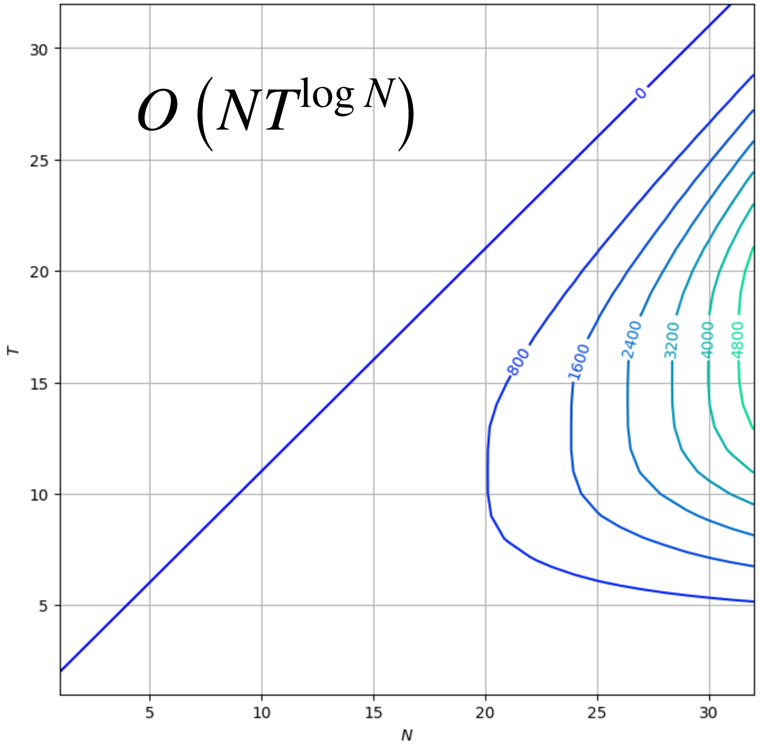

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  - Active security with ZKPs or with Secure Randomness

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Signing Complexity				
Share size				

	Shamir			
	Cyclic			
# Rounds	$2T + 1$			
Signing Complexity	$O(N\ \lambda^2)$			
Share size	$O(1)$			

	Shamir	Replicated		
	Cyclic	Non-Abelian		
# Rounds	2T + 1	$2\binom{N}{T-1}+1$		
Signing Complexity	O(N λ^2)	$O\left(\binom{N}{T-1}\lambda\right)$		
Share size	O(1)	$O\left(\binom{N}{T-1}\lambda\right)$		

	Shamir	Replicated	Vandermonde
	Cyclic	Non-Abelian	Non-Abelian
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Share size	O(1)	$O\left(\binom{N}{T-1}\lambda\right)$	$O(NT^{\log N}\lambda)$

	Shamir	Replicated	Vandermonde	Vandermonde
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- **Option 1**: CSI-SharK (1) introduces PVP,

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- **Option 3** *[OPEN]*: can we have DKG for the Vandermonde Sharing?

Thanks