A Montgomery ladder for isogenies

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SQIparty

30 April 2025

CSIDH

Private	Public	Private
	E_0	
Alice		Bob

CSIDH

Private	Public	Private
	E_0	
$[\mathfrak{a}]$		$[\mathfrak{b}]$
Alice		Bob

CSIDH

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$[\mathfrak{a}]$		$[\mathfrak{b}]$
Alice	$\xrightarrow{[\mathfrak{a}]E_0}$	Bob
	$\xleftarrow[{[\mathfrak{b}]E_0}]{}$	

CSIDH

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Alice	$\longrightarrow$ $\longleftarrow$	Bob
$[\mathfrak{a}][\mathfrak{b}]E_0$	$[\mathfrak{b}]E_0$	$[\mathfrak{b}][\mathfrak{a}]E_0$

Orientations

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Theorem

If the $\mathcal{O}$-orientation is primitive, this gives a free action

$$\mathrm{Cl}(\mathcal{O}) \circlearrowright \{(E, \iota)\}/\cong .$$

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Key exchange from a class group action

Private	Public	Private
	E_0	
$[\mathfrak{a}]$	$[\mathfrak{a}]E_0$	$[\mathfrak{b}]$
Alice	$\xrightarrow{\hspace{3em}}$ $\xleftarrow{\hspace{3em}}$	Bob
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(i) CRS

(ii) OSIDH

(iii) SCALLOP & friends

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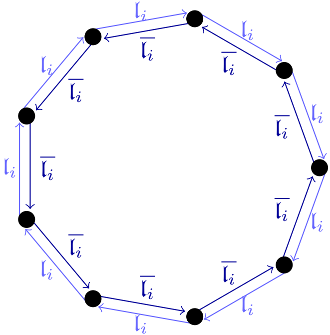
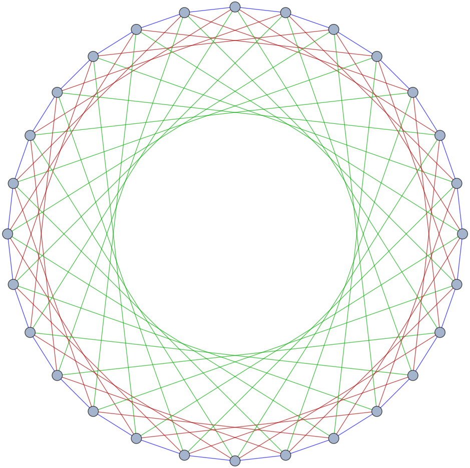


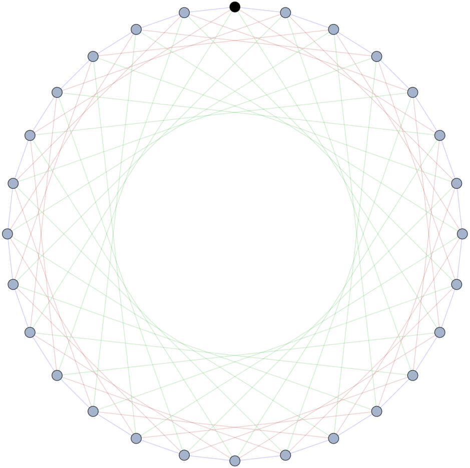
(Connected component of) the supersingular ℓ-isogeny graph over $\mathbb{F}_p$.

CSIDH



(Connected component of) a union of supersingular 3-, 5-, and 7-isogeny graphs over $\mathbb{F}_p$.

CSIDH



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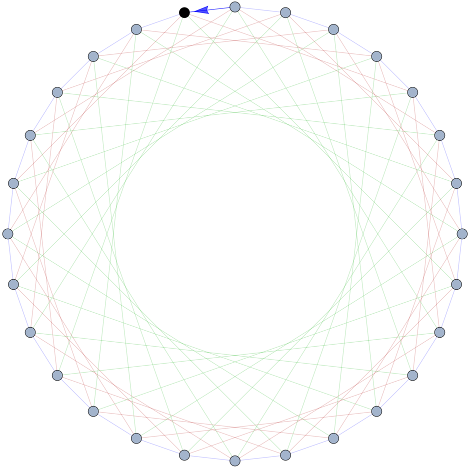

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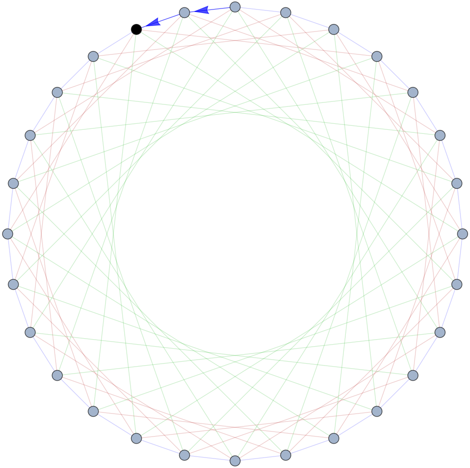

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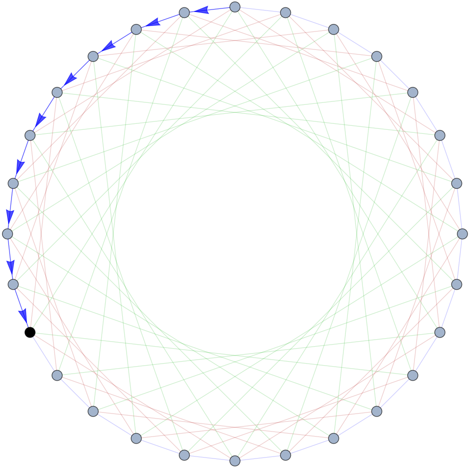

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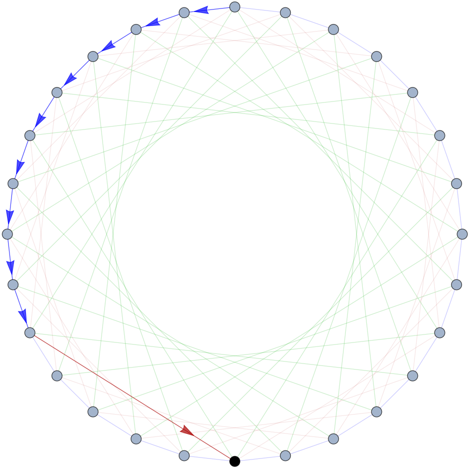

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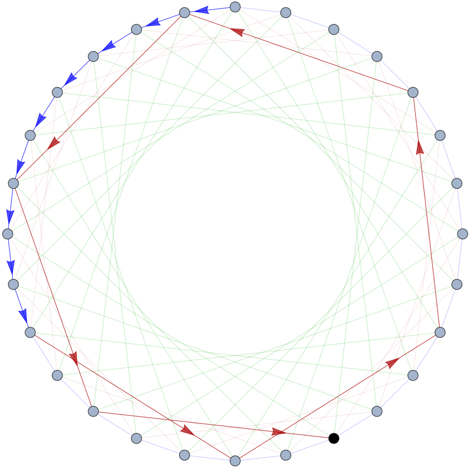

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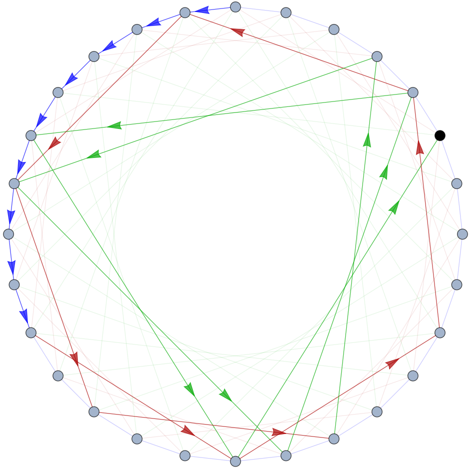

CSIDH-512

$$p = 4 \cdot \underbrace{(3 \cdot 5 \cdot \ldots \cdot 373)}_{73 \text{ consecutive primes}} \cdot 587 - 1 \approx 2^{511}.$$

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Computing the ℓ_i-isogenies

As $\mathcal{O} = \mathbb{Z}[\pi]$ -ideals

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Can compute action of $(a_1, \ldots, a_n) \in \{0, 1\}^n$ from one point $P \in E(\mathbb{F}_p) = E[\pi - 1]$ of order $\prod_{i=1}^{n} \ell_i$.

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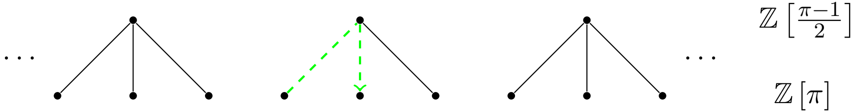

The supersingular $\mathbb{F}_p$-rational 2-isogeny graph if $p \equiv 3 \pmod 8$.

Moving to the surface

Assume $p = 8 \cdot \prod_{i=1}^{n} \ell_i - 1$.

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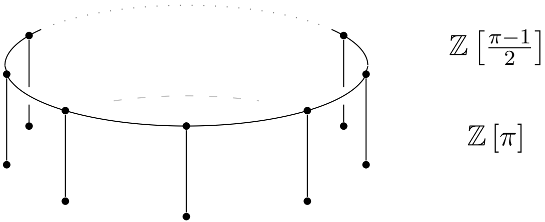
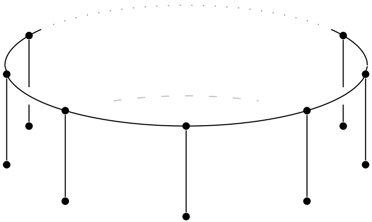


The supersingular $\mathbb{F}_p$-rational 2-isogeny graph if $p \equiv 7 \pmod 8$.

Moving to the surface

Assume $p = 8 \cdot \prod_{i=1}^{n} \ell_i - 1$.

$$\left(\frac{\pi - 1}{2}\right) = \left(2, \frac{\pi - 1}{2}\right) \cdot \prod_{i=1}^{n} \left(\ell_i, \frac{\pi - 1}{2}\right)$$



$\mathbb{Z}\left[\frac{\pi-1}{2}\right]$

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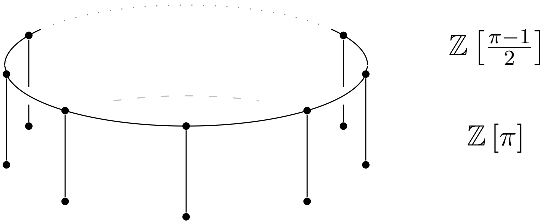

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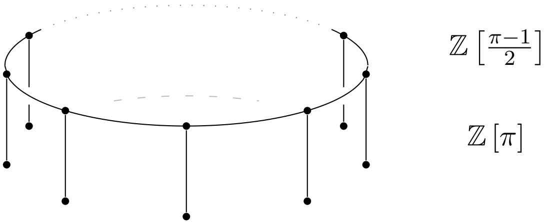

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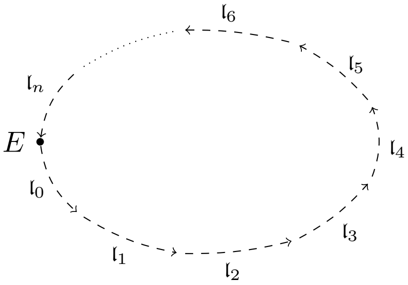
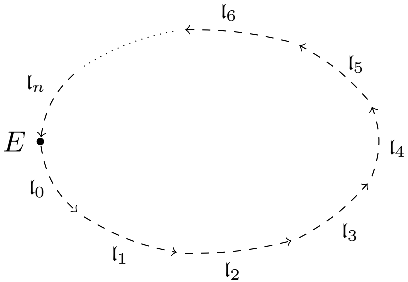
Acting by the trivial ideal class

$$\left(\frac{\pi - 1}{2}\right) = \prod_{i=0}^{n} \mathfrak{l}_i.$$



The action by the ideal class $(1, \ldots, 1)$.



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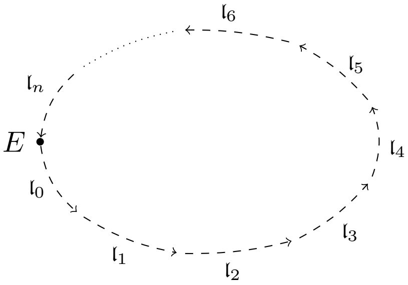

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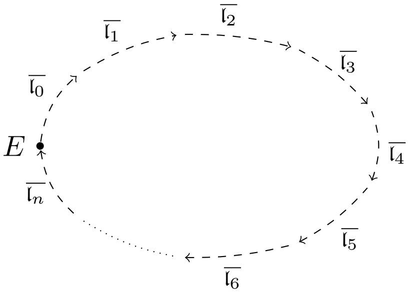

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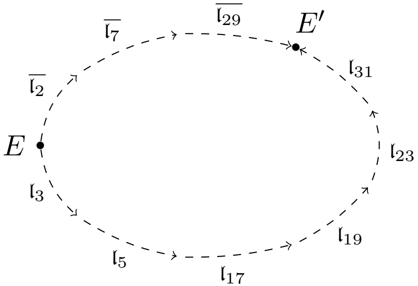
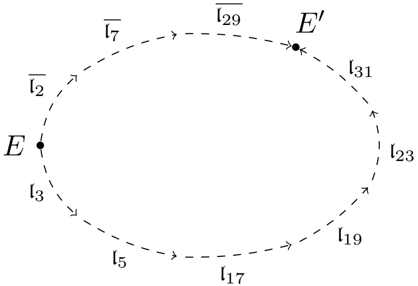


Acting by the ideal class $(0, 1, 1, 0, 1, 1, 1, 0, 1) \equiv (-1, 0, 0, -1, 0, 0, 0, -1, 0)$.



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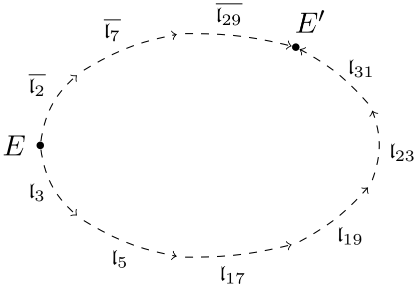

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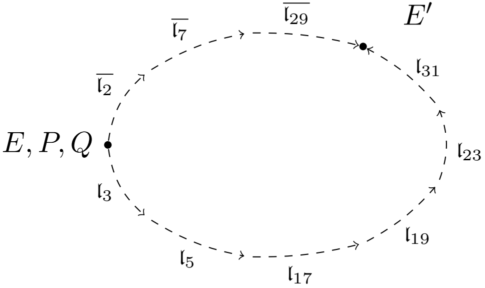
and

$$\varphi^- : E \to E/\langle [3\cdot 5\cdot 17\cdot 19\cdot 23\cdot 31]Q\rangle \cong E', \quad \text{where} \quad E\left[\frac{\pi+1}{2}\right] = \langle Q\rangle.$$


$\overline{\mathfrak{l}_{29}}$
E'
$\overline{\mathfrak{l}_7}$
$\mathfrak{l}_{31}$
$\overline{\mathfrak{l}_2}$
E, P, Q
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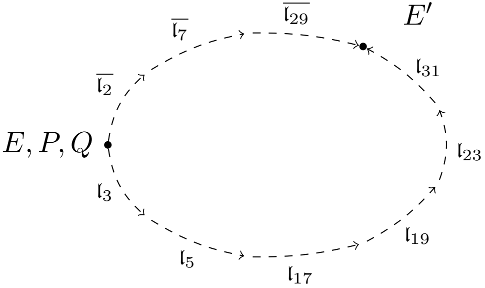
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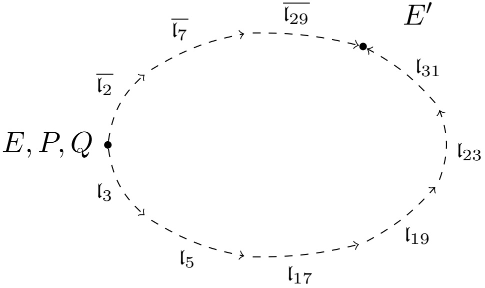
Magic™



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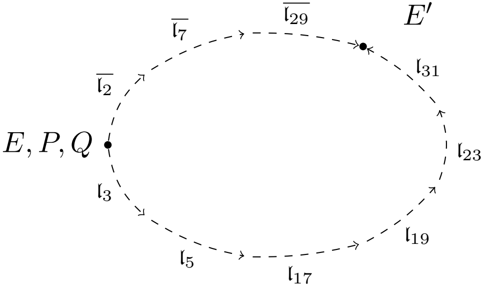

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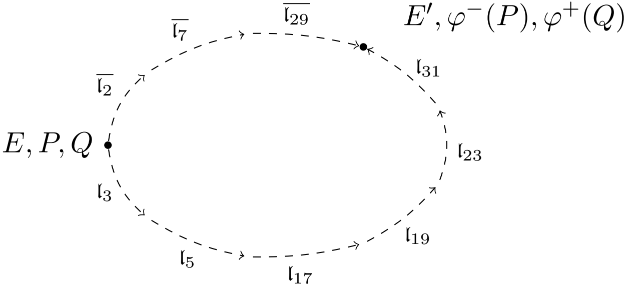

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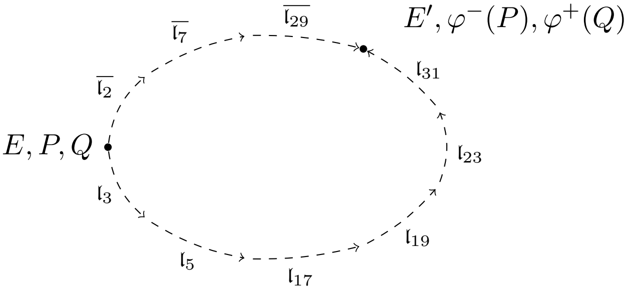

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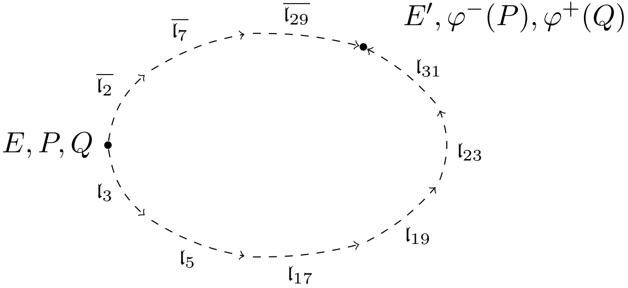
$\Longrightarrow$ iterate to apply the action by any ideal class $(a_1, \ldots, a_n) \in \mathbb{Z}^n$.



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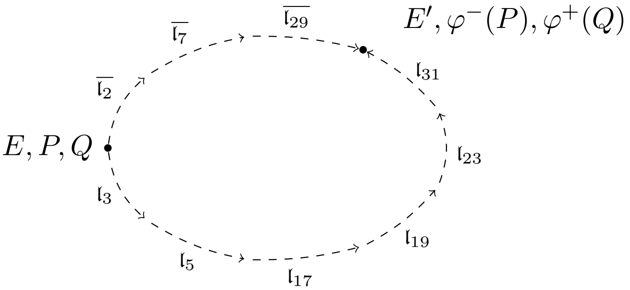
One ℓ_i-isogeny for every i (i.e. one evaluation of $\frac{\pi-1}{2}$).



"Montgomery ladder" for binary ideal classes

$R_0 \leftarrow (E, P, Q), R_1 \leftarrow (E, Q, P)$;
for $i = 0 \ldots n$ **do**
  $\mathsf{cswap}(R_0, R_1, \neg\mathsf{sk}[i])$;
  $R_0 \leftarrow \mathsf{Isogeny}(R_0, \ell_i)$; ▷ Compute ℓ_i-isogeny from $R_0[1]$; push $R_0[1], R_0[2]$.
  $R_1 \leftarrow \mathsf{Multiply}(R_1, \ell_i)$; ▷ Multiply $R_1[1]$ by ℓ_i.
end for;
$R_0[1] \leftarrow R_1[2]$;
return R_0;

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$$\left(\frac{\pi-1}{2}\right)=\prod\left(\ell_i,\frac{\pi-1}{2}\right)=\prod\mathfrak{l}_i.$$

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If $N(\sigma) = \prod \ell_i^{e_i}$, then (assume $\gcd(N(\sigma), \mathrm{Disc}(\mathcal{O})) = 1$)

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Quantum security

Depends on $\#\mathrm{Cl}(\mathcal{O}) \approx 0.46 |\mathrm{Disc}(\mathcal{O})|^{1/2}$.

CSIDH parameter estimates

Recent estimates of p for various NIST levels[1], based on SQALE[2].

Prime bits	f	n	Excluded	Included	Key Space	NIST level
p2048	2^{64}	226	$\{1361\}$	–	2^{221}	1 (aggressive)
p4096	2^{1728}	262	$\{347\}$	$\{1699\}$	2^{256}	1 (conservative)
p5120	2^{2944}	244	$\{227\}$	$\{1601\}$	2^{234}	2 (aggressive)
p6144	2^{3776}	262	$\{283\}$	$\{1693, 1697, 1741\}$	2^{256}	2 (conservative)
p8192	2^{4992}	338	$\{401\}$	$\{2287, 2377\}$	2^{332}	3 (aggressive)
p9216	2^{5440}	389	$\{179\}$	$\{2689, 2719\}$	2^{384}	3 (conservative)

[1] Campos, F., Chávez-Saab, J., Chi-Domínguez, J.J., Meyer, M., Reijnders, K., Rodríguez-Henríquez, F., Schwabe, P., Wiggers, T.: Optimizations and practicality of high-security CSIDH. CiC (2024).

[2] Chávez-Saab, J., Chi-Domínguez, J.J., Jaques, S., Rodríguez-Henríquez, F.: The SQALE of CSIDH: sublinear Vélu quantum-resistant isogeny action with low exponents. Journal of Cryptographic Engineering (2022).

Larger orientations

$$p + 1 = 4\prod \ell_i, \qquad (\sigma) = \prod (\ell_i, \sigma)^4 = \prod \mathfrak{l}_i^4.$$

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\bullet \arrow[d, dashed, "\varphi_4"'] & \bullet \arrow[l, dashed, "\varphi_3"'] \\
E_1 \arrow[r, dashed, "\varphi_1"'] & \bullet \arrow[u, dashed, "\varphi_2"']
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$\ker \varphi_1 = E_1 \left[\prod \mathfrak{l}_i\right] \subseteq \mathbb{F}_{p^2}.$

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E_1, P_1, Q_1 \arrow[r, dashed, "\varphi_1"'] & E_2, P_2, Q_2 \arrow[u, dashed, "\varphi_2"']
\end{tikzcd}$$

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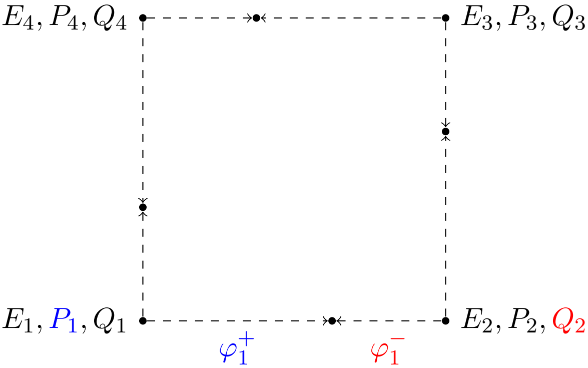
Acting by a binary ideal class



Example

$$\varphi_1^+ \leftrightarrow (1, 0, 1, 1, 0, \ldots), \qquad \varphi_1^- \leftrightarrow (0, -1, 0, 0, -1, \ldots).$$

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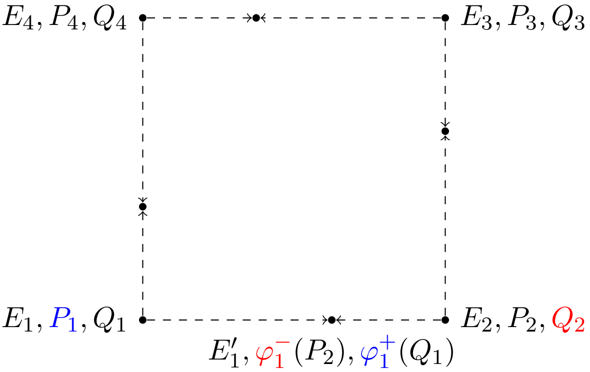

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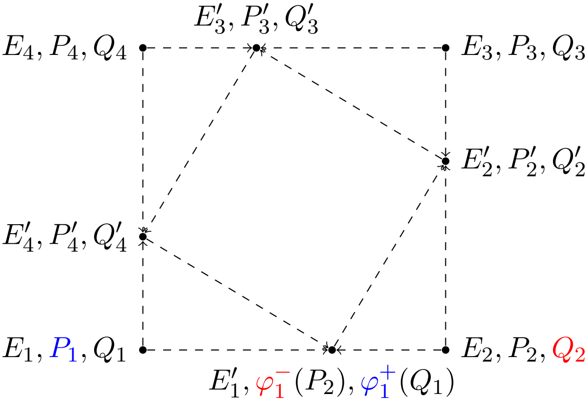

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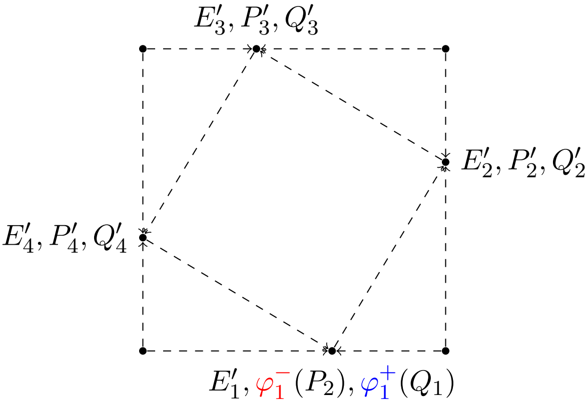

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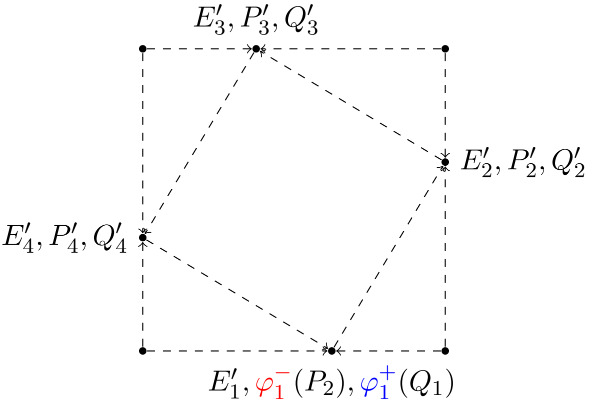

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Four ℓ_i-isogenies for every i, i.e. one evaluation of $\sigma = \prod \mathfrak{l}_i^4$.

Numbers

Let

$$p = 4 \cdot 3 \cdot 7 \cdot 13 \cdot 23 \cdot \underbrace{(3 \cdot 5 \cdot \ldots \cdot 293)}_{61 \text{ consecutive primes}} - 1 \cong 2^{413}.$$

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Then $E : y^2 = x^3 + x$ can be oriented by $\mathcal{O} = \mathbb{Z}[\sigma]$, where

$$N(\sigma) = \prod_i \ell_i^{5e_i}, \qquad \mathrm{tr}(\sigma) = 1130299,$$

such that

$$\mathrm{Disc}(\sigma) \cong 2^{2058} \text{ is prime.}$$

More numbers

Let

$$p = 4 \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 17 \cdot \underbrace{(3 \cdot 5 \cdot \ldots \cdot 337)}_{67 \text{ consecutive primes}} - 1 \cong 2^{457}.$$

Then $E : y^2 = x^3 + x$ can be oriented by $\mathcal{O} = \mathbb{Z}[\sigma]$, where

$$N(\sigma) = \prod_i \ell_i^{9e_i}, \qquad \operatorname{tr}(\sigma) = 3672029,$$

such that

$$\operatorname{Disc}(\sigma) \cong 2^{4100} \text{ is prime.}$$

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(iv) In particular, there exist families of class group action-based NIKEs more efficient than CSIDH (at a given NIST security level).

Thank you!

Algorithm 1 Evaluating a class group action using two kernel points

Input: An elliptic curve E/k, generators $P \in E[\sigma], Q \in E[\hat{\sigma}]$, a vector of integers $(s_1, \ldots, s_n) \in [0, e_i]^n$.

Output: The curve $E' := [\prod_i \mathfrak{l}_i^{s_i}] * E$, generators $P' \in E'[\sigma], Q' \in E'[\hat{\sigma}]$.

$(E^+, P^+, Q') \leftarrow (E, P, Q)$; ▷ $P^+ \in E^+[\sigma]$ and $Q' \in E^+[\hat{\sigma}]$.
$(E^-, P^-, P') \leftarrow (E, Q, P)$; ▷ $P^- \in E^-[\hat{\sigma}]$ and $P' \in E^-[\sigma]$.
$m \leftarrow \prod_i \ell_i^{e_i}$;
for $i = 1, \ldots, n$ **do**
  for $j = 1, \ldots, e_i$ **do**
    if $j \leq s_i$ **then**
      $m \leftarrow m/\ell_i$; $K \leftarrow [m]P^+$; ▷ K has order ℓ_i.
      $(E^+, P^+, Q') \leftarrow$ EvalEllIsogeny(E^+, K, P^+, Q'); ▷ "Isogeny"
      $P^- \leftarrow [\ell_i]P^-$; ▷ "Multiply"
    else ▷ Same as above, but with the roles of E^+ and E^- swapped.
      $m \leftarrow m/\ell_i$; $K \leftarrow [m]P^-$;
      $(E^-, P^-, P') \leftarrow$ EvalEllIsogeny(E^-, K, P^-, P');
      $P^+ \leftarrow [\ell_i]P^+$;
    end if
  end for
end for
assert $E^+ = E^-$;
return (E^+, P', Q');