# On prime degree twisting endomorphisms of supersingular elliptic curves

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# Index

# Elliptic curves, j-invariant, Frobenius

$$p \text{ any prime} > 3, q = p^n, a, b \in \mathbb{F}_q$$

$$E : y^2 = x^3 + ax + b, \quad \mathbf{j_E} = 1728\frac{4a^3}{4a^3 + 27b^2} \in \mathbb{F}_q$$

$$E^p : y^2 = x^3 + a^p x + b^p, \quad {j_E}^p = j_{E^p}$$

$$\phi_p : E(\overline{\mathbb{F}}_q) \longrightarrow E^p(\overline{\mathbb{F}}_q) \quad \text{If } a, b \in \mathbb{F}_p \text{ then } \phi_p = \pi_E \in End_{\mathbb{F}_p}(E)$$

$$(x, y) \longmapsto (x^p, y^p)$$

**Example** : $p \equiv 2 \mod 3, E : y^2 = x^3 + 1_{|\mathbb{F}_p}, j_E = 0$

$1 \neq$ cube $\mod p$. If $\xi \in \mathbb{F}_{p^2}$ with $\xi^2 + \xi + 1 = 0$ then $\xi^p = \xi^2$

$\Rightarrow \xi(x, y) = (\xi x, y) \in End(E)$ satisfies $\pi_E \xi = \xi^2 \pi_E \Rightarrow E$ supersingular

# Isomorphisms and Quadratic Twists

$$\textbf{Isomorphisms}: E^{'} : Y^2 = X^3 + a^{'}X + b^{'}, \quad a^{'}, b^{'} \in \mathbb{F}_q$$

$$E^{'} \in Isom_k(E) \Longleftrightarrow (X, Y) = (v^2x, v^3y) \text{ for } v \in k \mid a^{'} = av^4,\ b^{'} = bv^6$$

$$\textbf{Twists}: E^{t} : Y^2 = X^3 + au^2X + bu^3{}_{|\mathbb{F}_p}$$

$$E^{t} \in Twist(E) \Longleftrightarrow E^{t} \in Isom_k(E),\ k \text{ extension of } \mathbb{F}_q. \text{Then } j_E = j_{E^t}.$$

*Beware* : ([CPV, Lemma 1]) if $p \equiv 3 \mod 4$ and $b = 0$ and $u \in \mathbb{F}_p$

$$\Longrightarrow E^{t} \in Isom_{\mathbb{F}_p}(E) \text{ regardless of} \left(\frac{u}{p}\right) = \pm 1$$

because $u^2$ is always a square of $\mathbb{F}_p \Longrightarrow u^2 = v^4$ for some $v \in \mathbb{F}_p$

$$\Longrightarrow (X, Y) = (v^2x, v^3y) \text{ isomorphism defined over } \mathbb{F}_p$$

Non trivial twist: $\omega \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p \mid \omega^2 = u,\ E^{t} : Y^2 = X^3 + auX_{|\mathbb{F}_p}$

isomorphism $(X, Y) = (ux, u\omega y)$ now defined over $\mathbb{F}_{p^2}$

# j-invariant 1728

$$\textbf{Example :} \quad p \equiv 3 \mod 4$$

$$E_1 : y^2 = x^3 + x_{|\mathbb{F}_p}, \quad E_2 : y^2 = x^3 - x_{|\mathbb{F}_p}, \quad j_{E_1} = j_{E_2} = 1728$$

- $E_1, E_2$ are non-trivial quadratic twists one of each other
- $E_1^t = E_2$ with an isomorphism

$$(X, Y) = (ix, i(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})y)$$

- $E_1, E_2$ are 2-isogenous: $E_1 \in Isom_{\mathbb{F}_p}(E_2/[2]E_2)$
- If $j = 1728$ then $E_1 \xrightarrow{\varphi} {E_1}^t$, $\varphi$ isogeny with coefficients in $\mathbb{F}_p$

# j-invariant 1728

**Example** (*continued*) : $p \equiv 3 \mod 4$

$$E_1 : y^2 = x^3 + x_{|\mathbb{F}_p}, \quad E_2 : y^2 = x^3 - x_{|\mathbb{F}_p}, \quad j_{E_1} = j_{E_2} = 1728$$

- $i^p = -i$ so $\iota(x, y) = (-x, iy) \in End(E_1) \cap End(E_2)$ satisfies

$$\pi_E \iota = -\iota \pi_E$$

- Hence $E_1$ and $E_2$ are supersingular over $\mathbb{F}_p$
- Is $\iota$ related to $E_1 \xrightarrow{\varphi_{|\mathbb{F}_p}} {E_1}^t$?
- Is this situation holding for other $j_E$ ? We follow [CPV, Sect. 3] and ask for other $E$ and $\alpha \in End(E)$ such that

$$\pi_E \circ \alpha = -\alpha \circ \pi_E$$

# Quotient isogenies

- $\mathcal{I}_G : E \to E/G$ defined over $\mathbb{F}_p$ if and only if $\pi_E(G) \subseteq G$.
- $G_1, \ldots, G_{\ell+1}$ the $\ell + 1$ subgroups of order $\ell$ in $E(\overline{\mathbb{F}}_q)$
- $E/G_i$ the $\ell + 1$ curves adjacent to $E$ by $\ell$-isogenies $\mathcal{I}_{G_i}$ with coefficients in any extension of $\mathbb{F}_q$
- $\Phi_\ell(X, Y)$ the classical modular polynomial w.r.t. $\ell$
- $j(E/G_1), \ldots, j(E/G_{\ell+1})$ are roots of $\Phi_\ell(X, j(E))$.
- Solution of $\Phi_\ell(X, X) \mod p = 0$ represent $\overline{\mathbb{F}}_q$-isomorphism classes of curves with the same $j$-invariant ( so twists ) with isogenies of degree $\ell$ and coefficients over any extension.
- Our algorithm identifies the roots of $\Phi_\ell(X, X) \mod p$ where the "1728 scenario $E \xrightarrow{\varphi_{|\mathbb{F}_p}} E^t$ " takes place.

# 1728 scenario for other $j$

## Proposition

*If $E_{|\mathbb{F}_p}$ with $j(E) \neq 1728$, and $E^t$ non trivial $E^t \in Twist(E)$ s.t. $\varphi_{|\mathbb{F}_p} : E \to E^t$ isogeny over $\mathbb{F}_p$, then $\exists\, \alpha \in End(E)$ such that*

$$\pi_E \circ \alpha = -\alpha \circ \pi_E$$

$$\varphi_{|\mathbb{F}_p} \Rightarrow E^t_{|\mathbb{F}_p}$$

$$E^t \text{non trivial} \Rightarrow \exists u \notin \mathbb{F}_p \mid (X, Y) = (u^2 x, u^3 y),\ a' = u^4 a,\ b' = u^6 b$$

$$\Rightarrow u^2 \in \mathbb{F}_p \Rightarrow (u^2)^{(p-1)/2} = -1 \Rightarrow u^p = (u^2)^{(p-1)/2} u = -u$$

# twisting endos

$$\begin{array}{ccccc}
E & \xrightarrow{\varphi} & E^t & \xrightarrow{\psi_u} & E \\
\downarrow \pi_E & & \downarrow \pi_{E^t} & & \downarrow \pi_E \\
E & \xrightarrow[\varphi]{} & E^t & \xrightarrow[-\psi_u]{} & E
\end{array}$$

$$\alpha = \psi_u \circ \varphi \Longrightarrow \pi_E \circ \alpha = -\alpha \circ \pi_E$$

# Isogenies not defined in $\mathbb{F}_p$ come in pairs

$$\mathcal{Q}(E) = \{\mathcal{I}_{\langle P\rangle} : E \longrightarrow E/\langle P\rangle \mid \pi_E(P) \notin \langle P\rangle, ord(P) = \ell\}$$

$$\mathcal{T}(E, C) = \{\mathcal{I} : E \to E' \mid \mathcal{I} \in \mathcal{Q}(E), E' \in Twist(C)\}$$

**Lemma**

*If $E, C$ supersingular over $\mathbb{F}_p$ then $\#\mathcal{T}(E, C)$ is even.*

Let $P_1 \in E(\overline{\mathbb{F}}_p)$ of order $\ell$ such that $\mathcal{I}_{\langle P_1\rangle} \in \mathcal{Q}(E)$. Assume

$$C \cong E/\langle P_1\rangle = E_1$$

Then $j_{E_1} \in \mathbb{F}_p$. We will find another isogeny $\in \mathcal{Q}(E)$ different from $\mathcal{I}_{\langle P_1\rangle}$.

# Isogenies not defined in $\mathbb{F}_p$ come in pairs

Let $P_2 = \pi_E(P_1)$, $E_2 = E/\langle P_2\rangle$. Then $E_2 = {E_1}^p$.

But $j(E_1) \in \mathbb{F}_p$, so $j(E_2) = j({E_1}^p) = j(E_1)^p = j(E_1)$

$$\Rightarrow E_2 \in Twist(E_1)$$

$E$ supersingular $\Rightarrow \pi_E^2 = -[p]_E$

Hence $\pi_E(P_2) = \pi_E^2(P_1) \in \langle P_1\rangle \notin \langle P_2\rangle$

Therefore $\mathcal{I}_{\langle P_2\rangle} : E \to E_2 \in \mathcal{T}(E, C)$

# $\mathcal{G}_i(\mathbb{F}_p, \ell)$

- If $p \equiv 3 \mod 4$ then $K = \mathbb{Q}(\sqrt{-p})$ has discriminant $d_K = -p$
- $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{-p}}{2}]$, $\mathbb{Z}[\pi_E] \subseteq \mathcal{O}_K$ with index 2
- $$\mathbb{Z}[\pi] \subseteq End_{\mathbb{F}_p}(E) \subseteq \mathcal{O}_K$$
  one = one $\subset$ because of index constrain [K, DG]
- $\mathbb{Z}[\pi] = End_{\mathbb{F}_p}(E) \Longleftrightarrow E[2](\mathbb{F}_p)$ rank 1 [CPV, DG]
- Example $E_1, E_2$ with $j = 1728$.
- If $j_E \neq 1728$ then $End_{\mathbb{F}_p}(E) = End_{\mathbb{F}_p}(E^t)$ [ACLLNSS, Corollary 3.7]
- Write $O_2 = \mathbb{Z}[\pi_E]$ and $O_1 = \mathcal{O}_K$ and class numbers $h_1, h_2$

# $\mathcal{G}_i(\mathbb{F}_p, \ell)$

- The nodes of $\mathcal{G}(\overline{\mathbb{F}}_p, \ell)$ are $j$-invariants, and the arcs are $\ell$-isogenies defined over $\overline{\mathbb{F}}_p$
- The 1728 scenario $E \xrightarrow{\varphi_{|\mathbb{F}_p}} E^t$ becomes a node with a loop in $\mathcal{G}(\overline{\mathbb{F}}_p, \ell)$, hence a zero of $\Phi_\ell(X, X) \mod p$
- The 1728 scenario in $\mathcal{G}_i(\mathbb{F}_p, \ell)$ corresponds to

$$[\mathfrak{a}]^{-1}\mathcal{E} = ([\mathfrak{a}]\mathcal{E}^t)^t \text{ ([CPV, Lemma 5])}$$

- and an endomorphism $\alpha \in End(E)$ to a principal ideal in the maximal quaternionic order
- $\mathcal{G}_i(\mathbb{F}_p, \ell)$ :

nodes: $\mathcal{E}\ell\ell_p(O_i)$ $(Isom_{\mathbb{F}_p}(E) \mid End_{\mathbb{F}_p}(E) \cong O_i)$

arcs: $\ell$-isogenies $|_{\mathbb{F}_p}$

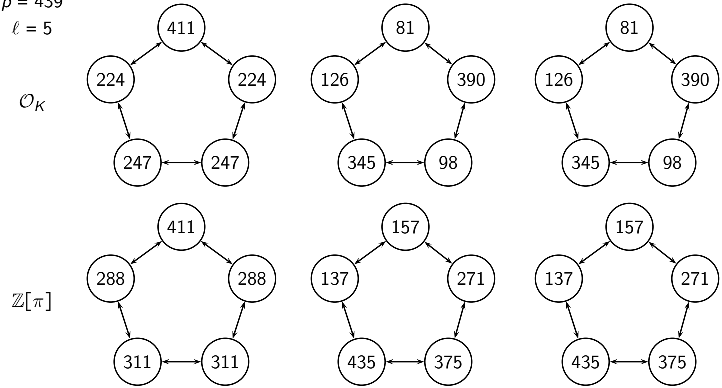
- Class group action on $\mathcal{E}\ell\ell_p(O_i)$ by ideal multiplication.

# $\mathcal{G}_i(\mathbb{F}_p, \ell)$

- $\ell \neq 2$ then a rational $\ell$-isogeny is ideal class of norm $\ell$ and order say $n_i$ in $\mathcal{C}\ell(O_i)$
- If $\left(\frac{-p}{\ell}\right) = 1$ then $(\ell)$ splits $\ell = \mathfrak{l}\bar{\mathfrak{l}}$ in $O_i$
- If also $\ell \neq 2$ and $p \equiv 3 \mod 4$ then $\mathcal{G}_1(\mathbb{F}_p, \ell)$, $\mathcal{G}_2(\mathbb{F}_p, \ell)$ are disconnected, so every node has just 2 horizontal arcs [K, DG]
- The $h_i$ classes in $\mathcal{G}_i(\mathbb{F}_p, \ell)$ are $\frac{h_i}{n_i}$ cycles of length $n_i$.
- If $p \equiv 3 \pmod 4$ then $h_i$ is odd, hence the length of both cycles $n_i$ is odd.
- If $n_i > 1$ then the 1728 scenario takes place for non principal ideals

# Example

$p = 439$
$\ell = 5$



# Algorithm

## Proposition

*If $p \equiv 3 \mod 4$ then the 1728 scenario $E \xrightarrow{\varphi_{|\mathbb{F}_p}} E^t$ takes place only in the cycle of $\mathcal{G}_i(\mathbb{F}_p, \ell)$ where $Isom_{\mathbb{F}_p}(C)$ with $j_C = 1728$ lies.*

Odd nodes + [CPV, Lemma 5]

## Proposition

*If $p \equiv 3 \mod 4$ and $E$ has a twisting endomorphism then the multiplicity of $j_E$ a zero of $\Phi_\ell(X, X) \mod p$ is odd.*

The 1728 scenario shows an arc joining $E$ and $E^t$ is $\varphi_{|\mathbb{F}_p}$. There might be more isogenies between them, but necessarily these are non-rational. By Lemma these come in pairs.

# Algorithm

## Proposition

- *If multiplicity of $j_E$ as a zero of $\Phi_\ell(X, X) \mod p$ is odd and $> 3$ then we found $j_E$ because then $E$ is necessarily supersingular.*
- *If multiplicity is 1 we have to sort supersingular $j$'s with factorization pattern of $\Phi_\ell(X, X) \mod p$ [BSS] (treat case $j = 0$ directly).*

# Algorithm

- set $c = 0$
- Find $\mathcal{O}_k, \mathbb{Z}[\pi]$ and find factorization of $\ell$ in both orders:
  1. $(\ell) = \mathfrak{e}_1\bar{\mathfrak{e}_1}$ in $O_1$
  2. $(\ell) = \mathfrak{e}_2\bar{\mathfrak{e}_2}$ in $O_2$
- If $\mathfrak{e}_1$ is ppal (in $O_1$) then add $j = 1728$ to list $J$
- If $\mathfrak{e}_2$ is ppal (in $O_2$) then add $j = 1728$ to $J$
- If both are ppal we are done

# Algorithm

- If $\mathfrak{c}_1$ is NOT ppal (in $O_1$) then all cycles have length $n_1 > 1$ and we are looking for a $j \neq 1728$
- The cycle $\mathcal{C}_1$ containing an isomorphism class with $j = 1728$ has the 1728 scenario. Put $c = c + 1$.
- If $\mathfrak{c}_2$ is NOT ppal (in $O_2$) then all cycles have length $n_2 > 1$ and we are looking for a $j \neq 1728$. Now $j$ can be $j = 0$.
- if $p \equiv 2 \mod 3$ and $\Phi_\ell(X, X) \mod p = X^k \cdot g(X) \mod p$ with $k$ odd then add $j = 0$ to the list
- else $j$ is $\neq 0$. Put $c = c + 1$

# Algorithm

- Set counter $r = 0$, and remove roots $0, 1728$ from $\Phi_\ell(X, X)$ mod $p$. Call it $f(X)$.
- While $r \neq c$ find root of $f(x)$ and check for odd multiplicity $m$.
- If $m \geq 3$ add the root to $J$ and set $r = r + 1$.
- If $m = 1$ the root may correspond to ordinary $j$ and rule out this case. If not, add the root to $J$ and $r = r + 1$
- return $J$

# More twisting endomorphisms

Examples given by our algorithm:

$$\mathbb{F}_{47}, \quad E: y^2 = x^3 + 16x + 15, \quad w^2 + 45w + 5 = 0$$

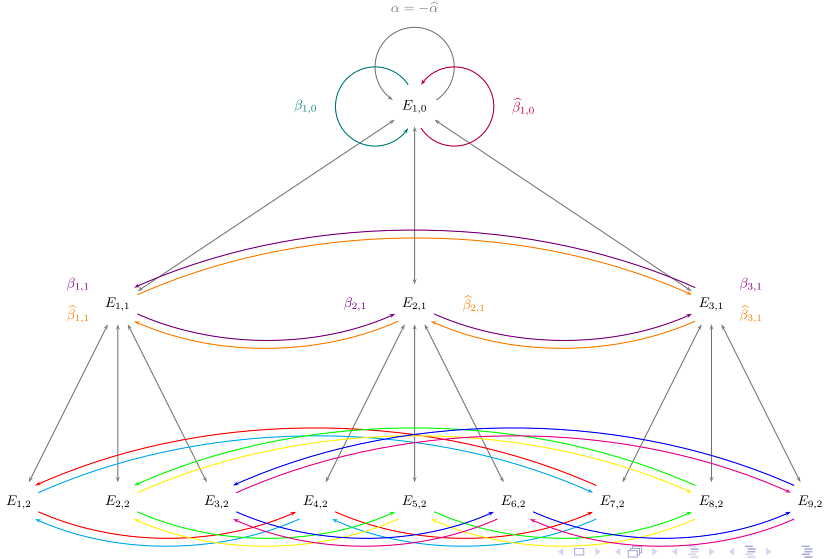
$$\alpha(x, y) = \Big(\frac{31x^3 + 7x^2 + 18x + 17}{x^2 + 26x + 28},$$

$$\frac{(32w + 15)x^3 + (26w + 21)x^2 + (w + 46)x + (23w + 24)}{x^3 + 39x^2 + 37x + 35}y\Big)$$

# Endomorphisms

- For $\alpha$ a twisting endomorphism of $E$ of order $\ell$ , let $\beta = r + s\alpha$ of prime degree $m = r^2 + s^2\ell$
- Then $\alpha$ spreads in the $\ell$-isogeny graph inducing endomorphisms of degree $m^{c_k}$ for $c_k \mid \ell^k$
- All elliptic curves $\ell^k$-isogenous to $E$ have an endomorphism of degree $m^{c_k}$

# Example $\ell = 3, r = 2, s = \pm 1, m = 7$



# Thank you!

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